Chapter 1
Axiomatizing Boolean Differentiation

Felix Weitkämper

Abstract In this contribution we give a complete axiomatization of Boolean differentiation. More precisely, for each $n \in \mathbb{N}$ and each Boolean algebra K we will state axioms that determine up to isomorphism the Boolean algebra with n linearly independent derivatives δ_i with $\bigcap_{i=1}^{n} \ker(\delta_i) \cong K$. Furthermore, we give a complete first-order theory T_n^K of n Boolean derivatives where $\bigcap_{i=1}^{n} \ker(\delta_i)$ is a model of the theory T_K. These theories can be obtained by adding just a finite list of axioms to those of T_K, and any model of T_K extends uniquely to a model of T_n^K. Moreover, we will also provide a theory of the additive reduct equipped with n Boolean derivatives, and we will see that these theories are categorical in every infinite cardinality. We then show that the theories are indeed the asymptotic theories of the class of the algebras of switching functions equipped with any of the ordinarily used notions of derivative. Furthermore, we see that for the case $n = 1$ they also axiomatize the Fraisse limit of the finite switching functions with a derivative, and we use this fact to deduce quantifier elimination.

1.1 Introduction

1.1.1 Our Approach

Derivative operations on Boolean algebras have been much studied since they were first described as such in the 1950s. An up-to-date textbook focused on the calculus as

Institut für Informatik
Ludwig-Maximilians-Universität München
Oettingenstr. 67
80538 München
felix.weitkaemper@lmu.de

well as on the numerous applications of Boolean differential operations is [12], while a concise systematic treatment of the calculus can be found in Chapter 10 of [11]. However, while algebraic and numeric aspects of differentiation on Boolean algebras have been widely studied, and various fields of application have been explored, to the best of our knowledge there has been no axiomatic investigation of Boolean differentiation since Kühnrich [6]. We will provide an axiomatic treatment that characterizes Boolean derivatives up to isomorphism. We will discuss how this can be adapted to an axiomatization of the first-order theory and outline some potential routes for further investigation and application. We will also see how the known notions of derviative fit into the framework we propose and clarify the relationship between our axioms and those of Kühnrich [6].

Another major motivation of this work comes from modern model theory, where over the last decades, two areas pertinent to this research have been explored in great depth:

Firstly, there is the model theory of difference fields, which are fields equipped with an automorphism. This has been developed extensively, using cutting-edge model-theoretic analysis such as the calculus of simple theories, and has found deep applications in number theory and algebraic dynamics (see [1] for an introduction). We will see later that, in fact, our setting is more aligned to that of difference algebra than to the setting of differential algebra that it is often compared with.

Secondly, there has been an upsurge in research on the connection between infinite models and their finite substructures, with conference volumes such as [4] dedicated to the topic and the monograph [2] summing up a whole line of research. This is particularly relevant here since most of the application interest lies in differentiation on finite rather than infinite Boolean algebras, while the power of model theoretic methods will be felt on the infinite level.

However, while we are inspired by the work on difference fields, our setting is quite different, since firstly we are entirely concerned with characteristic 2 (since $x + x = 0$ for the symmetric difference in a Boolean algebra) and secondly the automorphisms we study are involutions rather than free automorphisms. We will see that this combination will allow us to use a very small set of axioms compared to the axiomatizations of algebraically closed fields with an automorphism in [1]. This remains true even when we move towards several derivations, while the situation for several commuting automorphisms of fields is rather complicated.

One model-theoretic advantage of the field setting over that of Boolean algebras is that algebraically closed fields are *uncountably categorical*, while Boolean algebras are *unstable*. These classifications, which we will discuss briefly in Section 2 below, mean that the most powerful tools of contemporary model theory, those from stability theory, do not apply to Boolean algebras. Therefore, we will also present an axiomatization of the reduct to a language that contains purely the symmetric difference and the derivation(s). We will show that the theory of Boolean differentiation considered in this language is in fact *totally categorical*, a very strong model-theoretic property that means that it has a unique model up to isomorphism in every infinite cardinality. This allows the direct application of methods from [2], say, to our structures.

1.1.2 Applications of Axiomatizing Boolean Differentiation

We will briefly outline some potential ramifications of our different levels of axiomatization. Firstly, an immediate consequence of having an axiomatization *up to isomorphism* that applies to several known notions of derivative is that those notions are indeed isomorphic. This means that any property that is preserved under isomorphism transfers immediately from one derivative to another. For instance, we will see in Theorem 5 that both vectorial and simple derivatives of switching functions in the sense of Steinbach and Posthoff [12] fall under our axiomatization. Therefore, in order to prove something for all vectorial derivatives, it suffices to prove it for the simple derivative with respect to the first coordinate only, and it immediately generalizes to all the other notions of derivative.

From a structural point of view, a complete axiomatization lists essential properties of Boolean differentiation, since all properties of Boolean differentiation are bound to follow from the axioms. We will see that the axioms only need to pose a small number of algebraic conditions on the derivatives, which should sharpen the focus of further investigations into Boolean differentiation.

Having established an axiomatization up to isomorphism, what can be gained from an axiomatization of merely the first-order theory? One key advantage is that the first-order theory makes a connection between a single theory of derivatives on infinite Boolean algebras and the infinitely many theories of large finite Boolean Algebras with derivatives. The beginnings of this are developed in Subsection 1.4.

Beyond that, the connections to model theory and the work of Cherlin and Hrushovski [2] outlined above require a first-order axiomatization of the additive theory with derivatives. Exploiting this deeper connection of finite and infinite Boolean differentiation remains further work.

1.1.3 Outline

In the section following this introduction, we will be giving an overview of the terms and the results from model theory that we will be using in this paper.

In the main section, we will introduce Boolean differentiation and specifically our framework for derivatives. We will provide the axiom systems for the full language and the additive reduct and prove their completeness.

In Section 4 we will discuss the relationship to finite algebras of logic functions equipped with derivatives, and prove elimination of quantifiers for the theories with a single derivation.

In the final section we will discuss connections to the existing literature on Boolean differentiation. We will also highlight possible consequences of our results and point out some other putative areas for further research.

1.2 Model-theoretic Fundamentals

In this section we will rehearse the elements of classical model theory that we will need in the course of the paper. However, we will assume familiarity with the basic principles of first-order logic, such as its syntax and semantics, as well as fundamental concepts such as completeness of a theory and isomorphism of structures, which should be explained in any first textbook on logic.

Due to their traditional connection to propositional logic, Boolean algebras were among the first algebraic structures whose model theory was studied. In this work, we will refer to two classical complete theories of Boolean algebras: infinite atomic and infinite atomless Boolean algebras.

Proposition 1 *The following classes of Boolean algebras are axiomatizable by a complete first-order theory:*

1. *The theory of infinite atomless Boolean algebras;*
2. *The theory of infinite atomic Boolean algebras*

We will continue with some additional definitions:

Definition 1 A first-order theory is called *categorical* in a cardinal κ if all its models of cardinality κ are isomorphic.

It is well-known that the theory of infinite atomless Boolean algebras is ω-categorical, while the theory of infinite atomic Boolean algebras is not.

Categoricity is a central concept in model theory, as it implies both completeness and good model-theoretic behavior:

Proposition 2 *(Vaught's Test, Theorem 2.2.6 of [8]) If a satisfiable first-order theory with no finite models is categorical in an infinite cardinal κ, then it is complete.*

The most common measure of well-behavedness used in modern model theory is stability and the many variants of this concept, all of which have their root in Saharon Shelah's groundbreaking work on classification theory. We will refer to several steps on this scale, which we will briefly introduce here. We first need the concept of a type.

Definition 2 Let T be a complete theory, $\mathbb{M}$ a model of T, A a subset of $\mathbb{M}$ and $n \in \mathbb{N}$. Then a *(complete) n-type p of T over A* is a set of formulas with n free variables and parameters in A such that p is satisfiable and for every such formula ϕ, either ϕ or $\neg\phi$ lies in p.

The number of types that are realized in a certain model is at the basis of one of a number of equivalent definitions of stability. However, since we will need a different formulation later, we will give that here:

Definition 3 Let T be a complete theory in a countable language.

T is called *stable* if no formula has the order property: that is, there is no model $\mathbb{M}$ of T and formula $\phi(x; y)$ such that for a sequence of pairs of tuples $(a_i; b_i)_{i<\omega}$ in $\mathbb{M}$, $\phi(a_i; b_j)$ holds if and only if $i < j$.

T is called *ω-stable* if there are only countably many types over any countable subset of a model of *T*

T is called *strongly minimal* if every definable subset of any model of *T* is either finite or cofinite (i.e. its complement is finite).

These categories of stability are related to another in a strictly descending scale as follows:

Proposition 3 *For complete theories T in a countable language, the following strict implications hold: (i) ⇒(ii)⇒(iii)⇒(iv), where*

(i)	*T is strongly minimal.*
(ii)	*T is categorical in one (equivalently all) uncountable cardinal(s).*
(iii)	*T is ω-stable.*
(iv)	*T is stable.*

Proof (i) implies (ii) by Proposition 6.1.12, (ii) implies (iii) by Theorem 5.2.10, (iii) implies (iv) by Proposition 6.2.11 with Theorem 6.2.14, all from [8]. □

One of the prime reasons for the usefulness of stability theory is its connection to the existence of a good dimension notion on all models of the theory. The most commonly used and strongest dimension notion is known as *Morley Rank (*alongside the associated notion of a *Morley Degree)* and is usually abbreviated as *RM*. While one can find a rigorous introduction of the notion in Chapter 6 of [8], we will here just note the relationship between the existence of a well-defined Morley Rank and the stability hierarchy given above:

Proposition 4 *Let T be a complete theory in a countable language.*

Then T is strongly minimal if and only if every model of T has Morley Rank 1 and Morley Degree 1.

If T is uncountably categorical, every model has finite Morley Rank.

T is ω-stable if and only if (every definable subset of) every model has well-defined Morley Rank.

Remark 1 T being just stable is characterized by a different, but less well-behaved rank notion being well-defined.

We will conclude our excursion to stability theory by applying the stability hierarchy to Boolean algebras.

Proposition 5 *Let T be a theory that interprets an infinite Boolean algebra. Then T is unstable.*

Proof The canonical order relation of any infinite Boolean algebra, given by $a \leq b$ iff $a = a \wedge b$, has the order property in the sense of Definition 3. □

Therefore, we will not just study the full theory of the differential Boolean calculus, but also its reduct to the additive group of the associated Boolean ring. That is an abelian group with $x + x = 0$ for all x, and thus an $\mathbb{F}_2$-vector-space. This reduct is on the opposite end of the stability spectrum:

Proposition 6 *The theory of infinite abelian groups with* $x+x = 0$ *for all* x *is strongly minimal.*

Proof Classical result of model theory, see e.g. Section 4.5 of [5]. □

We will now continue to those concepts that help to characterize the relationship between finite and infinite structures.

First, we will introduce the concept of a generic theory, specialized to a context appropriate for our investigations:

Definition 4 Let $\mathcal{L}$ be a language and let $(\mathbb{M}_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{L}$-structures. Let T be a complete $\mathcal{L}$-theory.

T is called the *generic theory* of $(\mathbb{M}_n)_{n\in\mathbb{N}}$ if for all $\varphi \in T$ there is an $N \in \mathbb{N}$ such that $\mathbb{M}_n \models \varphi$ for all $n > N$.

Since all finite Boolean algebras are atomic, the generic theory of the cardinality-ascending sequence of finite Boolean algebras is the theory of infinite atomic Boolean algebras.

A generic theory can be considered as a limit of the individual theories of a sequence of structures.

A different notion which may or may not coincide with a generic theory can be obtained by turning this around and considering instead the first-order theory of the limit of the structures.

The notion of limit used here is the Fraisse limit of structures, for which there are different formalizations in slightly different settings. For our purposes, we will need one that can accommodate functions as well as relations, and we find it in Section 7 of [5].

Definition 5 Let $\mathbb{M}$ be an $\mathcal{L}$-structure. Then

$\mathbb{M}$ is called *locally finite* if any finitely generated substructure of $\mathbb{M}$ is finite.

A locally finite structure is called *uniformly locally finite* if there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that the substructure generated by any subset of cardinality n has cardinality at most $f(n)$.

A locally finite $\mathbb{M}$ is called *ultrahomogeneous* if every isomorphism between finite substructures extends to an isomorphism of $\mathbb{M}$.

If $\mathbb{M}$ is countably infinite, ultrahomogeneous and locally finite, it is referred to as a *Fraisse structure.*

Such a Fraisse structure is considered the *Fraisse limit* of the class of its finite substructures.

Proposition 7 *(Theorem 7.1.2 of [5]) A non-empty class of finite structures* $\mathfrak{K}$ *is the class of finite substructures of a Fraisse structure (i.e. has a Fraisse limit) if the following are satisfied:*

1. $\mathfrak{K}$ is closed under isomorphism.
   a. $\mathfrak{K}$ is closed under taking substructures.
   b. $\mathfrak{K}$ contains structures of arbitrarily large cardinalities.

c. Whenever A and B are in $\mathfrak{K}$, there is a C in $\mathfrak{K}$ such that both A and B can be embedded in C (Joint embedding property).
d. Whenever A, B_1 and B_2 are in $\mathfrak{K}$, $f_1 : A \to B_1$ and $f_2 : A \to B_2$, there are a $C \in \mathfrak{K}$ and embeddings $g_1 : B_1 \to C$ and $g_2 : B_2 \to C$ such that $g_1 \circ f_1 = g_2 \circ f_2$ (Amalgamation property).

Sometimes the generic theory of a class $\mathfrak{K}$ and the theory of the Fraisse limit coincide. For instance, the theory of infinite $\mathbb{F}_2$-vector-spaces is both the generic theory and the theory of the Fraisse limit of the class of finite $\mathbb{F}_2$-vector-spaces. For Boolean algebras, however, both notions of limit exist, but they do not coincide: While the generic theory of the class of finite Boolean algebras is the theory of infinite atomic Boolean algebras, their Fraisse limit is atomless (Classical, see e.g. Example 6.5.25 of [9]).

A very useful consequence of ultrahomogeneity is that the theory of a Fraisse structure will often be ω-categorical and admit quantifier elimination:

Proposition 8 *(Theorem 7.4.1 of [5]) Let $\mathbb{M}$ be a uniformly locally finite Fraisse structure. Then the theory of $\mathbb{M}$ is ω-categorical and admits quantifier elimination.*

As both abelian groups with $x + x = 0$ for all x and Boolean algebras are uniformly locally finite, the theory of atomless Boolean algebras and the theory of infinite abelian groups with $x + x = 0$ are ω-categorical and admit quantifier elimination.

1.3 Axiomatizing Boolean Differentiation

1.3.1 Boolean Functions, Rings and Derivations

The first prerequisite for a study of structures endowed with derivative operations is to recognize the underlying algebraic nature of those structures.

We will formulate this paper entirely in the context of *Boolean rings*, which is equivalent to that of Boolean algebras.

Definition 6 A *Boolean ring* $(\mathbb{B}, +, \cdot, 0, 1)$ is a commutative ring with unit that satisfies the following properties

1. *Idempotency*: For any $x \in \mathbb{B}$, $x \cdot x = x$.
2. *Characteristic 2*: For any $x \in \mathbb{B}$, $x + x = 0$.

Any Boolean algebra can be made into a Boolean ring by treating + as the symmetric difference (sometimes written $\oplus$ to avoid ambiguity) and $\cdot$ as the conjunction. Conversely, any Boolean ring defines a Boolean algebra, with conjunction taken as $\cdot$, disjunction as $x + y + xy$ and negation as $x + 1$. See [10] for the details.

This representation suits our purposes very well, since derivations are usually defined using the symmetric difference.

The most used derivations arise in the study of *switching functions*, that is, functions from $\{0, 1\}^n \to \{0, 1\}$ for an $n \in \mathbb{N}$. We will now formally introduce these derivations:

Definition 7 Let $n \in \mathbb{N}$, and let $f : \{0,1\}^n \to \{0,1\}$.

Then the *derivative of f with respect to the i-th coordinate $\delta_i(f)$* is given by the function

$$\delta_i(f) : \{0,1\}^n \to \{0,1\}\,,$$
$$\delta_i(f)(a_1,\ldots,a_i,\ldots,a_n) := f(a_1,\ldots,a_i,\ldots,a_n) + f(a_1,\ldots,a_i',\ldots,a_n).$$

The *global derivative $D(f)$* is given by $D(f)(x) = D(f)(x')$.

These derivatives have been extensively studied, and are the topic of the recent monograph [12]. In that and other work, a generalized notion of derivative that the authors call *vectorial derivative* is also introduced.

Definition 8 Let $n \in \mathbb{N}$, $f : \{0,1\}^n \to \{0,1\}$ and let $S \subseteq \{1,\ldots,n\}$. Then the *vectorial derivative of f with respect to S, $\delta_S(f)$,* is given by the function

$$\delta_S(f) : \{0,1\}^n \to \{0,1\}\,,\ \delta_S(f)(a_1,\ldots,a_n) := f(a_1,\ldots,a_n) + f(b_1,\ldots,b_n),$$

where $b_i = \begin{cases} a_i' & i \in S \\ a_i & i \notin S \end{cases}$.

In the literature, Boolean differentiation is studied mainly as an analogue to real or complex differentiation, and its algebraic properties are usually considered analogues to real or complex *differential* algebra (a remarkable exception to this being [13]).

However, while the above-mentioned derivatives are additive and factor over constants (i.e. functions whose derivative is 0), they do not satisfy the Leibniz rule of differentiation, that is, $\delta(xy) = x\delta(y) + y\delta(x) + \delta(x)\delta(y)$ rather than $\delta(xy) = x\delta(y) + y\delta(x)$, and indeed no possible notion of derivative could satisfy the classical definition of a derivation (cf. [11], Ch. 10).

In this paper, we will instead consider Boolean differentiation as an analogue of classical *difference* algebra, which studies automorphisms of the real or complex field. This possibility arises from the following observation:

Proposition 9 *In all of the cases above, the map $f \to \sigma(f)$, $\sigma(f)(a_1,\ldots,a_n) := f(b_1,\ldots,b_n)$, is an involution of the Boolean ring of functions from $\{0,1\}^n \to \{0,1\}$.*

Proof We need to show that σ respects addition and multiplication and that $\sigma^2 = \sigma$.

1. $\sigma(f+g)(a_1,\ldots,a_n) = (f+g)(b_1,\ldots,b_n) = f(b_1,\ldots,b_n) + g(b_1,\ldots,b_n) = \sigma(f)(a_1,\ldots,a_n) + \sigma(b)(a_1,\ldots,a_n)$
2. Similarly for multiplication
3. $\sigma^2(f)(a_1,\ldots,a_n) = f(b_1,\ldots,b_n)$, where $b_i = \begin{cases} a_i'' & i \in S \\ a_i & i \notin S \end{cases}$.

   But as $a_i'' = a_i$, $b_i = a_i$ for all i. □

As $\delta(f) = f + \sigma(f)$, $\sigma(f) = f + \delta(f)$ and we can (and will) therefore study derivations and their associated involutions interchangeably.

1.3.2 A Complete Axiomatization

In the light of Proposition 9, we can choose between using a derivation δ or an involution σ in our language, and whether to include the operations of a Boolean algebra or the ring operations. For the sake of consistency with the notion of Boolean *differentiation*, we will officially present our axiomatization in the following languages:

Definition 9 For $n \in \mathbb{N}$, let $\mathcal{L}_n$ be the language consisting of the binary operations $+$ and $\cdot$, the constant symbols 0 and 1 and the unary functions $\delta_1, \ldots, \delta_n$.

Let $\mathcal{L}_n^+$ be the reduct of this language, where the conjunction $\cdot$ and the constant 1 are omitted.

Whenever R is an $\mathcal{L}_n$ or $\mathcal{L}_n^+$ structure, let $\sigma_n := \delta_n + \mathrm{id}$.

In $\mathcal{L}_1$ and $\mathcal{L}_1^+$, we usually write δ and σ for δ_1 and σ_1.

For clarity of exposition, we will begin by providing a complete axiomatization of the Boolean derivative on $\mathcal{L}_n^+$ and then extending it to a complete axiomatization on $\mathcal{L}_n$.

Definition 10 Let T_1^+ be the following $\mathcal{L}_1^+$ theory:

1. V is an abelian group of characteristic 2, that is, an abelian group with the property that $\forall x(x + x = 0)$.
2. σ is an involution of groups.
3. δ is complete, that is, $\forall y(\delta(y) = 0 \Rightarrow \exists x(\delta(x) = y))$.

We will not only show that T_1^+ is complete when restricted to infinite models, but moreover, we will show that it is categorical in every infinite cardinal:

Theorem 1 *T_1^+ is categorical in all infinite cardinals. Its infinite models form a complete ω-stable elementary class.*

The proof of Theorem 1 will go through two Lemmas. First, though, a simple observation that we will use throughout and which justifies the formulation of the completeness axiom:

Remark 2 Let V be an abelian group of characteristic 2 and σ an involution of groups. Then $\forall x \in V : \delta(\delta(x)) = 0$.

Proof $\delta(\delta(x)) = \delta(x+\sigma(x)) = (x+\sigma(x))+\sigma(x+\sigma(x)) = x+\sigma(x)+\sigma(x)+x = 0.$□

Lemma 1 *Let V, V' be free finite-dimensional k-modules over a ring k and let $F : V \to V'$ be a linear isomorphism. Let $f : V \to V$ and $f' : V' \to V'$ be linear endomorphisms. Let* **a** *be a basis for V and M a matrix representing f with respect to* **a**. *Then F is an isomorphism of the structures enriched by a function symbol for f on V and f' on V' iff M is the matrix representation of f' with respect to $\overrightarrow{F(a)}$.*

Proof It suffices to show that $\forall x(F(f(x)) = f'(F(x)))$. So let $x \in V$ and let $\mathbf{v}$ be the k-vector representing x with respect to $\mathbf{a}$. Then $\mathbf{v}$ is also the k-vector representing $F(x)$ with respect to $\overrightarrow{F(a)}$. Thus

$$F(f(x)) = F(M\mathbf{va}) = M\mathbf{v}\overrightarrow{F(a)} = f'(\mathbf{v}\overrightarrow{F(a)}) = f'(F(\mathbf{va})) = f'(F(x)).$$

In order to apply Lemma 1 to our structures, we will prove another lemma.

Lemma 2 *Let $(V, +, 0, \delta)$ be a model of T_1^+. Then $(V, +, 0)$ is an $\mathbb{F}_2$ vector space and the following holds:*

1. *$V = \bigoplus_{i=1}^{\kappa} U_i$ for a cardinal κ, where each U_i is a 2-dimensional δ-invariant subspace on which δ can be represented by the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.*
2. *V has cardinality 2^{2n} for an $n \in \mathbb{N}$ or infinite cardinality.*

Proof The proof will proceed in steps.

First, as V is an abelian group, being of characteristic 2 is equivalent to being an $\mathbb{F}_2$ vector space.

Let K be the kernel of the group- and thus $\mathbb{F}_2$-vector-space-homomorphism δ. Let $(b_i | i \in I)$ be an $\mathbb{F}_2$-basis for K and let $(a_i | i \in I)$ be such that $\delta(a_i) = b_i$. We claim that $V = \bigoplus_{i \in I} \langle a_i, b_i \rangle$ is a decomposition as required in the statement of the lemma. So, we have to show (a) that $V = \sum_{i \in I} \langle a_i, b_i \rangle$, (b) that the sum is direct and (c) that each$\langle a_i, b_i \rangle$ satisfies the requirements of the lemma.

(a): Let $x \in V$. Then by Remark 2 $\delta(x) \in K$ and thus $\delta(x) = \sum_{j \in J} b_j$. Observe that $\delta(x + \sum_{j \in J} a_j) = \sum_{j \in J} b_j + \sum_{j \in J} b_j = 0$ and thus that $x + \sum_{j \in J} a_j \in K$. But as by definition $K \subseteq \sum_{i \in I} \langle a_i, b_i \rangle$ and $\sum_{j \in J} a_j \in \sum_{i \in I} \langle a_i, b_i \rangle$, we also obtain $x \in \sum_{i \in I} \langle a_i, b_i \rangle$.

(b): We need to show that $\sum_{j \in J} u_j = 0 \Rightarrow u_j = 0$ for all $j \in J$. But by definition $\sum_{j \in J} u_j = \sum_{k \in K} a_k + \sum_{l \in L} b_l$. We see that $\delta(\sum_{k \in K} a_k + \sum_{l \in L} b_l) = \sum_{k \in K} b_k$ and since $(b_i | i \in I)$ is a basis for K, this implies that $K = \emptyset$. Then $\sum_{l \in L} b_l = 0$, which however implies that $L = \emptyset$ by the same argument.

(c): We have already seen that each $\langle a_i, b_i \rangle$ is 2-dimensional, so it remains to show that $\delta(a_i) = b_i$ and that $\delta(b_i) = 0$. But that is just the definition of the a_i and b_i.

This shows the first clause of the Lemma; the second clause follows from the first clause together with additivity of dimension in free sums and the fact that $|V| = 2^{\dim_{\mathbb{F}_2}(V)}$. □

Remark 3 In fact, one can extend any linearly independent system $\mathbf{w_i}$ in the kernel together with any $\mathbf{v_i}$ with $\delta(v_i) = w_i$ into a representation with respect to which the lemma holds.

We can now proceed to prove Theorem 1.

Proof Let $(V, +, \delta)$ and $(V', +, \delta')$ be two models of T_1^+ of cardinality $\kappa \geq \omega$. Then by Lemma 2, $V = \bigoplus_{i=1}^{\kappa} U_i$ and $V' = \bigoplus_{i=1}^{\kappa} U_i'$ with the properties mentioned there. We define a linear bijection $F : V \longrightarrow V'$ by defining linear bijections $F_i : U_i \longrightarrow U_i'$ for each i. Let (a_i, b_i) and (a_i', b_i') be bases for U_i and U_i' respectively for which δ has the matrix representation $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then let $F_i(a_i) = a_i'$ and $F_i(b_i) = b_i'$. Clearly, F_i defines an isomorphism of vector spaces, and by Lemma 1, $F_i(\delta(x)) = \delta'(F(x))$. We will now define $F(x) = F(\sum u_j) := \sum F_j(u_j)$. This is clearly a well-defined linear bijection. It thus only remains to show that $F(\delta(x)) = \delta'(F(x))$:

$$F(\delta(x)) = \sum F_j(\delta(u_j)) = \sum \delta'(F_j(u_j)) = \delta'(\sum F_j(u_j)) = \delta'(F(x)).$$

Therefore, T_1^+ is categorical in all infinite cardinals. By the discussion in Section 1.2, this implies that the first-order theory of the infinite models of T_1^+ is both complete and ω-stable (since it is uncountably categorical).

This categoricity result unlocks powerful model-theoretic tools for Boolean differential groups, which we will briefly discuss in the final section. Here we will now adapt our axiomatization to give a complete first-order theory of Boolean differentiation which takes full account of the ring structure. □

Definition 11 Let K be a Boolean ring, and T_K a complete first-order theory of Boolean algebras expressed in the language of Boolean rings. Then T_1^K is the following theory in the language $\mathcal{L}_1$:

1. σ is an involution of Boolean rings.
2. $\ker(\delta) \models T_K$.
3. δ is complete, i.e. there is a $z \in V$ such that $\delta(z) = 1$.

Remark 4 We remark that we found it rather surprising that one could obtain a complete axiomatization by just adding a finite number of axioms to the ones regarding K. This seems to be entirely due to the fact that one can define the ring structure on V from the ring structure on the constants (see below).

We will adopt a different and possibly more straightforward strategy to proving completeness of the first-order theory here, extending isomorphisms between kernels to isomorphisms between the models of T_1^K. First, we give a more concrete characterization of δ being complete:

Proposition 10 *Let V be a model of T_1^K for a Boolean ring K. Then V is a free* $\ker(\delta)$*-algebra on two generators* $(1, z)$ *and δ is a* $\ker(\delta)$*-algebra-morphism given by* $\delta(z) = 1$ *and* $\delta(1) = 0$.

Proof Let z be as in the definition of T_1^K.

(a) $(1, z)$ generate V. Indeed, let $x \in V$ be arbitrary. Then $\delta(x) \in \ker(\delta)$ and $\delta(x) = \delta(\delta(x)z)$ by $\ker(\delta)$-linearity. Thus, $x + \delta(x)z \in \ker(\delta)$ and therefore $x = (x + \delta(x)z) + \delta(x)z$ is the required representation.

(b) $(1, z)$ generate V freely. Indeed, if $a + bz = 0$ for some $a, b \in K$, then $\delta(a + bz) = b = 0$ and thus also $a = 0$. □

Now we can prove the extension of isomorphisms.

Proposition 11 *There is a one-to-one correspondence between isomorphism classes of Boolean algebras K and isomorphism classes of models of* T_1^K.

Proof Let K be a Boolean algebra and V a free K-algebra on 2 generators. Then by Lemma 1, the condition $\delta(z) = 1$ and $\delta(1) = 0$ uniquely determines V as a K-algebra up to isomorphism.

So let $f : V \longrightarrow V'$ be an isomorphism of K-algebras respecting δ. We claim that f is in fact an isomorphism of Boolean rings. So let $(k_1 + k_2 z)$ and $(k_1' + k_2' z)$ be elements of V. Then

$$\begin{aligned}
&f((k_1 + k_2 z) \cdot (k_1' + k_2' z)) = f(k_1 k_1' + (k_2 k_1' + k_1 k_2' + k_2 k_2')z) \\
&= f(k_1)f(k_1') + (f(k_2)f(k_1') + f(k_1)f(k_2') + f(k_2)f(k_2'))f(z) \\
&= f(k_1 + k_2 z)f(k_1' + k_2' z)
\end{aligned}$$

Therefore f is actually an isomorphism of Boolean rings as required. □

We can reformulate this as a complete axiomatization result in its own right:

Corollary 1 *For any Boolean algebra K, the following three axioms characterize a Boolean derivative with kernel K up to isomorphism:*

1. σ *is an involution of Boolean rings.*
2. $\ker(\delta) \cong K$.
3. δ *is complete, i.e. there is a* $z \in V$ *such that* $\delta(z) = 1$.

It follows from the above that whenever the theory of K is ω-categorical, then so is T_1^K. In particular, when K is an infinite atomless Boolean algebra, then T_1^K is ω-categorical and therefore complete. In fact, T_1^K is complete regardless of K, and this can be seen using any of a number of classical model-theoretic techniques.

Theorem 2 *Let* T_K *be any complete theory of Boolean rings. Then the theory* T_1^K *is complete.*

Proof We sketch a proof using ultraproducts (See Section 9.5 of [5] for an introduction), since that most easily generalizes to several derivations. Let A and B be models of T_1^K, and let K_A and K_B be their respective kernels. Then $K_A \equiv K_B$ and we want to show that $A \equiv B$ also. By the Keisler-Shelah Theorem, K_A and K_B have isomorphic ultrapowers $U(K_A) \simeq U(K_B)$. Using the same index set and the same ultrafilter, we can take the ultrapowers of $U(A)$ of A and $U(B)$ of B. Then the kernel of $U(A)$ is isomorphic to $U(K_A)$ and the kernel of $U(B)$ is isomorphic to $U(K_B)$. Thus, the kernels are isomorphic to each other and by Proposition 11 $U(A)$ and $U(B)$ are also. Therefore, A and B must have been elementarily equivalent. □

We will now extend the characterizations above to several derivatives.

In the following we will use the shorthand $\delta_J^{|J|}$ to mean the $|J|$-fold derivative with respect to all δ_j, $j \in J$; for instance, $\delta^n_{\{1,\ldots,n\}}(x) = \delta_1\delta_2 \ldots \delta_n(x)$ and $\delta^1_{\{j\}} = \delta_j$. (We add the cardinality superscript to avoid confusion with the vectorial derivative from Definition 8)

Definition 12 Let T_n^+ be the following $\mathcal{L}_n^+$ theory:

1. V is an abelian group of characteristic 2, that is, an abelian group with the property that $\forall x(x + x = 0)$.
2. $\sigma_1, \ldots, \sigma_n$ are commuting involutions of groups.
3. $\{\delta_1, \ldots, \delta_n\}$ is complete, that is,

$$\forall y(\delta_1(y) = 0 \wedge \delta_2(y) = 0 \wedge \ldots \wedge \delta_n(y) = 0 \Rightarrow \exists x(\delta_1\delta_2 \ldots \delta_n(x) = y)).$$

We will now provide an analogue to Lemma 2 to prove the categoricity of T_n^+ in each uncountable cardinal.

Lemma 3 *Let $(V, +, 0, \delta_1, \ldots, \delta_n)$ be a model of T_n^+. Then $(V, +, 0)$ is an $\mathbb{F}_2$ vector space and the following holds:*

1. *$V = \bigoplus_{i=1}^{\kappa} U_i$ for a cardinal κ, where each U_i is a 2^n-dimensional δ-invariant subspace which has a basis $(a_{i,J} | J \subseteq \{1, \ldots, n\})$ such that the following holds: $\{\langle\{a_{i,J}, a_{J\cup\{j\}}\}\rangle \,|\, j \notin J\}$ is a decomposition of U_i in the sense of Lemma 2 with respect to δ_j.*
2. *V has cardinality $2^{2^n m}$ for an $m \in \mathbb{N}$ or infinite cardinality.*

Proof The proof will proceed in steps.

First, as V is an abelian group, being of characteristic 2 is equivalent to being an $\mathbb{F}_2$ vector space.

Let (b_i) be a basis for $\bigcap_{j=1}^{n} K_i$, where $K_j := \ker(\delta_j)$. Then choose (a_i) such that $\delta_1\delta_2 \ldots \delta_n(a_i) = b_i$. Let $a_{i,J} := \delta_J^{|J|}(a_i)$. We claim that this satisfies the requirements, and we will prove this by induction. The case $n = 1$ has been shown in Lemma 2. So assume true for n. It is easy to see that K_{n+1} is a model of T_n^+. Therefore, by the induction hypothesis, $(a_{i,J} | J \subseteq \{1, \ldots, n+1\}, n+1 \in J)$ is a basis for K_{n+1} as required. But then by Lemma 2, $(a_{i,J} | J \subseteq \{1, \ldots, n\})$ is a basis of V with exactly the properties described in clause 1.

This shows the first clause of the Lemma; the second clause follows from the first clause together with additivity of dimension in free sums and the fact that $|V| = 2^{\dim_{\mathbb{F}_2}(V)}$. □

We can now deduce the completeness and indeed the total categoricity of T_n^+ just as we did for T_1^+:

Theorem 3 *T_n^+ is categorical in all infinite cardinals. Its infinite models form a complete ω-stable elementary class.*

Proof Just as in the proof of Theorem 1, the linear bijection induced by the bases given by Lemma 3 is an $\mathcal{L}_n$-isomorphism by Lemma 1. □

We will now finally provide an axiomatization of the complete theory of several derivations on Boolean rings:

Definition 13 Let K be a Boolean ring, and T_K a complete first-order theory of Boolean algebras expressed in the language of Boolean rings. Then T_n^K is the following theory in the language $\mathcal{L}_n$:

1. $\sigma_1, \ldots, \sigma_n$ are commuting involutions of Boolean rings.
2. $\bigcap_{i=1}^{n} \ker(\delta_i) \models T_K$.
3. $\{\delta_1, \ldots, \delta_n\}$ is complete, that is,

$$\exists x(\delta_1\delta_2 \ldots \delta_n(x) = 1)).$$

The proof will again be preceded by a proposition giving a more concrete representation.

Proposition 12 *Let V be a model of T_n^K for a Boolean ring K. Then V is a free $\bigcap_{i=1}^{n} \ker(\delta_i)$-algebra on 2^n generators given by $\{a_J := \delta_J^{|J|}(a) | J \subseteq \{1, \ldots, n\}\}$ for any $a \in V$ with $\delta_1\delta_2 \ldots \delta_n(a) = 1$.*

Proof By induction on n. The case $n = 1$ is part of Proposition 10. So assume it true for n and choose any model V of T_{n+1}^K and any $a \in V$ with $\delta_1\delta_2 \ldots \delta_{n+1}(a) = 1$.

We will now show that it is a generating system for V. So let $x \in V$. We will proceed by induction on the smallest number m such that the m-fold derivative $\delta_{\{1,\ldots,m\}}^m(x) \in \bigcap_{i=1}^{n} \ker(\delta_i)$. If $m = 0$ then $x \in \bigcap_{i=1}^{n} \ker(\delta_i)$ itself. So assume true for m. Then if $\delta_{\{1,\ldots,m+1\}}^m x \in \bigcap_{i=1}^{n} \ker(\delta_i)$, $x = (\delta_{\{1,\ldots,m+1\}}^m x) \cdot \delta_{\{1,\ldots,n\} \setminus \{1,\ldots,m+1\}}^{n-(m+1)} a + y$, $y := ((\delta_{\{1,\ldots,m+1\}}^m x)\delta_{\{1,\ldots,n\} \setminus \{1,\ldots,m+1\}}^{n-(m+1)} a + x)$. Here $\delta_{\{1,\ldots,m+1\}}^m y = 0$ and thus $\delta_{\{1,\ldots,m\}}^m y \in \bigcap_{i=1}^{m+1} \ker(\delta_i)$. □

Proposition 13 *Let K be a Boolean ring. Then there is exactly one model of T_K^n up to isomorphism with $\bigcap_{i=1}^{n} \ker(\delta_i) = K$.*

Proof We will prove the theorem by induction on n. The case $n = 1$ is exactly Proposition 11. So assume it true for T_K^n. We will now show it for T_K^{n+1}. Let a be the witness of clause 3. of the definition and let $a_J := \delta_J^{|J|}(a)$. Then we claim that the isomorphism of K-modules induced by a_J is an $\mathcal{L}_{n+1}$-isomorphism. By the induction hypothesis, it is an isomorphism of the obvious $\mathcal{L}_n$-structures on $K_i := \ker(\delta_i)$ for each derivation δ_i. However, since $\delta_i(a_{\{1,\ldots,i-1,i+1,\ldots,n\}}) = 1$, another application of Proposition 11 shows that we actually have an isomorphism of Boolean rings which also respects δ_i. Since i was arbitrarily chosen, this finishes the proof. □

Just as for T_1^K, we can reformulate this as an explicit axiomatization result and deduce completeness of the first-order theory:

Corollary 2 *For any Boolean algebra K, the following three axioms characterize Boolean derivatives with kernel K up to isomorphism:*

1. $\sigma_1, \ldots, \sigma_n$ *are commuting involutions of Boolean rings.*
2. $\bigcap_{i=1}^{n} \ker(\delta_i) \cong K$.
3. $\{\delta_1, \ldots, \delta_n\}$ *is complete, that is,*

$$\exists x(\delta_1\delta_2 \ldots \delta_n(x) = 1)).$$

Theorem 4 *Let T_K be any complete theory of Boolean rings, and let $n \in \mathbb{N}$. Then the theory T_n^K is complete.*

1.4 Relationship to Finite Models and Immediate Consequences

In this section we will be connecting the complete theories from Subsection 1.3.2 with the examples of Boolean differentiation studied in the literature.

In particular, we will show that the theories we have introduced can be naturally characterized as the generic or as the limit theories of groups or rings of switching functions equipped with the derivatives introduced in Subsection 1.3.1.

To facilitate notation, we introduce

Definition 14 Let $\mathbb{S}_n$ be the *Boolean ring of switching functions in n variables,* that is, the Boolean ring made up of all mappings $f : \{0, 1\}^n \to \{0, 1\}$, equipped with the ring structure from Subsection 1.3.1. Let $\mathbb{S}_n^+$ be the additive group reduct of $\mathbb{S}_n$.

Theorem 5 *The theory of infinite models of T_1^+ is the generic theory of the class $\{\mathbb{S}_n^+ | n \in \mathbb{N}\}$, where each switching algebra is equipped with any of the derivatives of Subsection 1.3.1.*

Proof By the results at the end of Subsection 1.3.1, each of the structures mentioned is a model of T_1^+. Clearly, $|\mathbb{S}_n| \geq n$ and thus the additional infinity axioms are generically true in the class too. So, the theory of infinite models of T_1^+ is a subset of the generic theory. However, as the theory is complete by Theorem 1, it is the generic theory. □

The equivalent result for the theory of Boolean rings is obtained in a very similar way; however, one has to choose a Boolean algebra that models the generic theory of finite Boolean algebras.

Theorem 6 *The theory T_1^K, where K is an infinite atomic Boolean algebra, is the generic theory of the class $\{\mathbb{S}_n | n \in \mathbb{N}\}$, where each switching algebra is equipped with any of the derivatives of Subsection1.3.1.*

Proof By the discussion following Definition 4, the theory of K is the generic theory of the class $\{\mathbb{S}_n | n \in \mathbb{N}\}$ as Boolean rings. Since $|\ker(\delta)| = \sqrt{|\mathbb{S}_n|}$ for all derivations mentioned in Subsection 1.3.1, the theory of $\ker(\delta)$ will indeed be generically T_K. The remainder of the axioms are clear. As T_1^K is complete by Theorem 11, we can conclude that T_1^K is the generic theory. □

The theorems above show that we have indeed given a characterization of the asymptotic theory of switching functions - so although our results and methods have focused on infinite models, they can be used to study the derivations on arbitrarily large finite switching algebras that have spawned such a large literature.

They also generalize to the theories with several derivations, when one considers derivations that are *linearly independent* in the sense of [12], but we will omit the generalization of the proofs here for brevity. One example of such linearly independent derivations are the single derivations $\delta_1, \ldots, \delta_n$ on $\mathbb{S}_n$. We have

Theorem 7 *The theory of infinite models of T_n^+ is the generic theory of the class $\{\mathbb{S}_i^+ | i \geq n\}$, where each switching algebra is equipped with the single derivatives δ_1 to δ_n.*

The theory T_n^K, where K is an infinite atomic Boolean algebra, is the generic theory of the class $\{\mathbb{S}_i | i \geq n\}$, where each switching algebra is equipped with the single derivatives δ_1 to δ_n.

We will now move on to characterize the theories we have constructed as the complete theories of limit structures. This gives us more information about their model theory and provides a concrete structure into which the finite switching algebras can be uniquely embedded up to isomorphism. We can take the limit over the same classes we have considered above. In the additive case, we will obtain exactly the same theory, as the underlying theory of infinite $\mathbb{F}_2$-vector spaces is both generic and limit theory of the finite $\mathbb{F}_2$-vector spaces. In the full Boolean ring case, however, we will have to change the Boolean ring K under consideration since the limit structure of finite Boolean rings is the countable atomless Boolean algebra and not a countable atomic Boolean algebra.

Theorem 8 *Let $\mathfrak{C}$ be the class of all substructures of a member of the class $\{\mathbb{S}_i^+ | i \in \mathbb{N}\}$, where each switching algebra is equipped with any of the derivatives of Subsection 1.3.1. Then $\mathfrak{C}$ is a Fraisse class and its limit structure is the unique countably infinite model of T_1^+.*

Proof We will go through the requirements of a Fraisse class one by one.

1. Closure under isomorphisms is clear.
2. Closure under substructure is guaranteed by our definition as being substructures of a certain other class of structures.
3. It contains arbitrarily large structures, as S_n lies in $\mathfrak{C}$.
4. We can always consider the larger of the two indices of the structures that they embed into.
5. Consider the situation of the amalgamation condition. As we can embed B_1 and B_2 into S_i and S_j respectively, we can assume without loss of generality that B_1 and B_2 are in $\{\mathbb{S}_i^+ | i \in \mathbb{N}\}$. Let δ denote the derivations. Without loss of generality, let the index of B_1 be at most the index of B_2. We will first build a basis for A, which we will then extend to bases for B_1 and B_2 in such a way that a natural embedding between the bases defines an embedding from B_1 into B_2. Start with a basis for $\delta(A) \subseteq A$. This can be extended to a basis for $\ker_A(\delta)$, and that in turn to bases $(\mathbf{k})$ and $(\mathbf{k'})$ of B_1 and B_2 respectively. By (the proof of) Lemma

2, **k** and **k′** together with any choice of preimages of **k** and **k′** define bases for B_1 and B_2. We can therefore choose the preimages in such a way that the preimage will be chosen from A wherever A contains such a preimage. We argue that the bases **b** and **b′** obtained in that way contain a basis for A. Indeed, by the standard kernel-image decomposition in linear algebra, the dimension of A is equal to the dimension of the image plus the dimension of the kernel, and the number of preimage elements that could be chosen from A is exactly the dimension of the image. So consider the embedding from B_1 into B_2 that is induced by mapping **b** to **b′** in an appropriate way. Then by Lemma 1, this is an isomorphism onto its image, i. e. an embedding, and it respects A as required by clause 5. □

Thus, we could also define the theory of infinite models of T_1^+ as the theory of the Fraisse limit of all finite switching algebras equipped with a derivation.

The analysis also yields quantifier elimination as a consequence:

Corollary 3 *The theory of infinite models of T_+^1 has quantifier elimination.*

Proof The substructure generated by a subset A of a differential group is the group generated by $A \cup \sigma(A)$. Thus, since the group reduct is uniformly locally finite, so is the Boolean differential group.

Thus, the result follows from Theorem 8 by Proposition 8. □

Considering the theory with the full Boolean algebra structure, we obtain a representation for T_1^K, where K is the countable atomless Boolean algebra.

Theorem 9 *Let $\mathfrak{C}$ be the class of all substructures of a member of the class $\{\mathbb{S}_i | i \in \mathbb{N}\}$, where each switching algebra is equipped with any of the derivatives of Subsection1.3.1. Then $\mathfrak{C}$ is a Fraisse class and its limit structure is the unique countably infinite model of T_1^K, where K is the countable atomless Boolean algebra.*

Proof As in the proof of Theorem 8, Clauses 1-4 are easily verified. We therefore consider the situation of the amalgamation property, and again we can assume without loss of generality that B_1 and B_2 are in $\{\mathbb{S}_i | i \in \mathbb{N}\}$ and that the index of B_1 is at most the index of B_2. Due to the corresponding property for pure Boolean algebras, we can furthermore assume that $\ker(\delta)_{B_1} \subseteq \ker(\delta)_{B_2}$ and that $(\ker(\delta) \cap A)_{B_1} = (\ker(\delta) \cap A)_{B_2}$. By the analysis in Chapter 3 of [12], $\delta(A)$ is itself a lattice of functions. In particular, $\delta(A)$ has a maximum, say $\alpha \in A$. Let $x \in A$ be chosen with $\delta(x) = \alpha$. Choose z_1 and z_2 in B_1 and B_2 respectively s. t. $\delta(z_1) = 1$ and $\delta(z_2) = 1$. We will define an embedding $\iota : B_1 \to B_2$ by setting ι to be the identity on $\ker(\delta)$ and choosing a value of $\iota(z_1)$. If $\delta(\iota(z_1)) = 1$, then ι is an embedding of Boolean differential algebras. So consider $x = \alpha z_1 + a$ in B_1 and $x = \alpha z_2 + b$ in B_2, where $\delta(a) = \delta(b) = 0$. Then we set $\iota(z_1) := z_2 + a + b$. This defines an embedding of Boolean differential algebras since $\delta(z_2 + a + b) = 1 + 0 + 0 = 1$. We thus have to show that for all $y \in A$, $\iota(y_{B_1}) = \iota(y_{B_2})$. First, we will see that this holds for x, and we will derive an auxiliary result:

$$\begin{aligned} x\alpha &= \alpha z_1 + a\alpha = x + (\alpha+1)a \\ &= \alpha z_2 + b\alpha = x + (\alpha+1)b \\ \Rightarrow& (\alpha+1)a = (\alpha+1)b \\ \Rightarrow& (\alpha+1)(a+b) = 0 \\ \Rightarrow& \alpha(a+b) = a+b \end{aligned}$$

So $\iota(x) = \alpha(z_2 + a + b) + a = \alpha z_2 + b$ as required. So now consider $y \in A$ arbitrary. Then $y = \beta z_1 + c = \beta\alpha z_1 + c = \beta(\alpha z_1 + a) + \beta a + c = \beta x + \beta a + c$. Since ι is the identity on the kernel elements β, a and c and $\iota(x_{B_1}) = \iota(x_{B_2})$ it follows that $\iota(y_{B_1}) = \iota(y_{B_2})$ as required.

Therefore the theory has a Fraisse limit.

Since ultrahomogeneity of the whole structure also implies ultrahomogeneity of the kernel, the kernel must be the countable atomless Boolean algebra. □

Just as for the additive theory, we can now conclude a quantifier elimination result:

Corollary 4 *Let K be an atomless Boolean algebra. Then the theory T_1^K admits quantifier elimination.*

Proof Just as Corollary 3 follows from Theorem 8. □

We will conclude this section by outlining how such results might be combined to investigate Boolean differentiation in large switching algebras. We will focus on the simplest case, T_+^1, for which Fraisse theory and generic theory coincide.

The approach rests on three results: By Theorem 5, the theory of infinite models of T_+^1 is the generic theory of finite switching algebras. By Corollary 3, this theory also eliminates quantifiers. Finally, since T_+^1 is finitely axiomatizable and the theory of its infinite models is complete, it is also decidable. We can put these results together and arrive at the following two-step procedure to decide whether a given tuple of switching functions in a large switching algebra satisfies any $\mathcal{L}_1^+$-formula $\varphi(\mathbf{x})$:

1. Find a a quantifier-free formula $\varphi_0(\mathbf{x})$ such that $\varphi_0(\mathbf{x})$ is equivalent to $\varphi(\mathbf{x})$ on all infinite models of T_+^1. Since their theory is decidable, $\varphi_0(\mathbf{x})$ can be determined effectively.
2. Check whether the given tuple of switching functions $\mathbf{f}$ satisfies $\varphi_0(\mathbf{x})$. Since the theory of infinite models of T_+^1 is generic, $\varphi_0(\mathbf{f})$ iff $\varphi(\mathbf{f})$ whenever $\mathbf{f}$ is from a sufficiently large switching algebra.

Since quantifier-free formulas do not reference any other objects of the algebra, it can be checked without regard to the switching algebra from which $\mathbf{f}$ is taken but merely by inspecting $\mathbf{f}$ itself.

1.5 Future Applications and Perspectives

In this section we will briefly discuss the connection between the first-order theory as presented here and Kühnrich's abstract notion of a Boolean derivative (see Chapter 10 of [11]). We will then explore potential applications and directions for further research.

While this is to the best of our knowledge the first analysis of the first-order theory of Boolean differentiation or of axioms complete up to isomorphism, there has certainly been some work on a more general framework for the different notions of derivative suggested in the literature. One such framework, which has been proposed by Martin Kühnrich ([6]), is presented in the chapter on Boolean differentiation in [11]:

Definition 15 Let B be a Boolean ring and let $d : B \to B$. Then d is called a *(Kühnrich) differential operator* if the following hold:

1. For all $x \in B$, $d(d(x)) = 0$.
2. For all $x \in B$, $d(x + 1) = dx$.
3. For all $x, y \in B$, $d(xy) = xd(y) + yd(x) + d(x)d(y)$.

Since Kühnrich's axioms do not include any notion of completeness, they are essentially weaker than the theory presented here. In fact, Kühnrich's differential operator has a simple characterization in terms of involutions:

Proposition 14 *Let B be a Boolean ring and let $d : B \to B$. Then d is a (Kühnrich) differential operator if and only if $\sigma : B \to B$, $\sigma(x) = x + d(x)$, is an involution of Boolean rings.*

Proof "⇒": We will verify that σ respects addition, multiplication, 0 and 1.

1. d respects addition by Proposition 10.2.1 of [11]. Thus $\sigma(x+y) = x+y+d(x+y) = x + d(x) + y + d(y) = \sigma(x) + \sigma(y)$.
2. $\sigma(xy) = xy + (xd(y) + yd(x) + d(x)d(y)) = (x + d(x))(y + d(y)) = \sigma(x)\sigma(y)$.
3. $d(0) = d(1) = 0$ by Proposition 10.2.1 of [11]. Thus $\sigma(0) = 0 + 0 = 1$ and $\sigma(1) = 0 + 1 = 1$. □

"⇐": Let σ be an involution and $d(x) := x+\sigma(x)$. We will verify Kühnrich's axioms for d.

1.

$$d(d(x)) = d(x)+\sigma(d(x)) = x+\sigma(x)+\sigma(x)+\sigma(\sigma(x)) = x+\sigma(x)+\sigma(x)+x = 0.$$

2.

$$d(x + 1) = x + 1 + \sigma(x) + \sigma(1) = x + \sigma(x) = d(x).$$

3.

$$\begin{aligned} xd(y) + yd(x) + d(x)d(y) =& x(y + \sigma(y)) + y(x + \sigma(x)) + (x + \sigma(x))(y + \sigma(y)) \\ =& xy + x\sigma(y) + xy + y\sigma(x) + xy + x\sigma(y) + y\sigma(x) + \sigma(x)\sigma(y) \\ =& xy + \sigma(x)\sigma(y) = d(xy). \end{aligned}$$

This holds completely analogously for the "Boolean differential algebras of order k" that are introduced in Definition 10.2.2 of [11]; they are exactly characterized by inducing k commuting involutions of B.

This characterization suggests the question of the exact relationship between Kühnrich's operators and the models of the theories introduced here. In particular, it is clear that every substructure of a model of T_1^K (T_n^K) is a differential operator (algebra) in this sense. But does the converse hold? By Corollary 6.5.3 of [5], this is equivalent to the question of whether Kühnrich's axioms axiomatize the universal theory of T_1^K (T_n^K).

A particular interest lies in the connections between the finite structures that are studied in the literature and the complete theories of infinite structures expounded here. For the particular case of the additive reduct, this is especially alluring, since their infinite models form a totally categorical theory. The connection between totally categorical theories and their finite substructures is the subject of a deep model-theoretic analysis around so-called *smoothly approximable structures* as discussed for instance in [7] and [2]. In particular, the Morley rank of a definable set in the theory of infinite models determines the approximate size of the respective definable subset of a finite model in a precise and uniform manner.

Of course, stability theory brings a host of inter-related concepts in its own right too, and investigating these notions with respect to additive Boolean differentiation would be an important contribution towards bringing the theory of difference algebra in the Boolean case to a similar level as the more widely studied difference algebra over fields.

Furthermore, it would be very interesting to extend T_n^K and T_n^+ to countably infinitely many derivations. Then, one would have one single theory encompassing switching functions of arbitrary sizes and their derivatives. Using more sophisticated model-theoretic techniques, one might also be able to extend the stability hierarchy in order to adequately cover this case.

The quantifier elimination results in Section 4 beg the question to what extent they can be extended to the theories with several derivatives. It would also be interesting to consider how quantifier elimination results for other theories of Boolean algebras, that might require additional predicates, can be extended to quantifier elimination results for the corresponding theory T_1^K. One example is the theory of infinite atomic Boolean algebras, which admits quantifier elimination if one adds predicates for "n atoms lie below x" (see [3] for details and further examples).

References

[1] Zoé Chatzidakis. "Model theory of difference fields". In: *The Notre Dame lectures*. Vol. 18. Lect. Notes Log. Assoc. Symbol. Logic, Urbana, IL, 2005, pp. 45–96.

[2] Gregory Cherlin and Ehud Hrushovski. *Finite Structures with Few types*. Annals of Mathematics Studies 152. Princeton UP, 2003.

[3] Jamshid Derakhshan and Angus Macintyre. "Enrichments of Boolean algebras by Presburger predicates". In: *Fund. Math.* 239.1 (2017), pp. 1–17. ISSN: 0016-2736.
[4] Javier Esparza, Christian Michaux, and Charles Steinhorn, eds. *Finite and algorithmic model theory*. Vol. 379. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2011, pp. xii+341. ISBN: 978-0-521-71820-2.
[5] Wilfrid Hodges. *Model theory*. Vol. 42. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1993, pp. xiv+772. ISBN: 0-521-30442-3.
[6] Martin Kühnrich. "Differentialoperatoren über Booleschen Algebren". In: *Z. Math. Logik Grundlag. Math.* 32.3 (1986), pp. 271–288. ISSN: 0044-3050.
[7] Dugald Macpherson and Charles Steinhorn. "Definability in classes of finite structures". In: *Finite and algorithmic model theory*. Vol. 379. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2011, pp. 140–176.
[8] David Marker. *Model theory*. Vol. 217. Graduate Texts in Mathematics. An introduction. Springer-Verlag, New York, 2002, pp. viii+342. ISBN: 0-387-98760-6.
[9] Vladimir Pestov. *Dynamics of infinite-dimensional groups*. Vol. 40. University Lecture Series. American Mathematical Society, Providence, RI, 2006, pp. viii+192. ISBN: 978-0-8218-4137-2; 0-8218-4137-8.
[10] Sergiu Rudeanu. *Boolean Functions and Equations*. North Holland Publishing Company, Amsterdam, 1974.
[11] Sergiu Rudeanu. *Lattice Function and Equations*. Springer, 2001.
[12] Bernd Steinbach and Christian Posthoff. *Boolean Differential Calculus*. Synthesis Lectures on Digital Circuits and Systems 52. Morgan & Claypool, 2017.
[13] André Thayse. *Boolean calculus of differences*. Vol. 101. Lecture Notes in Computer Science. With a foreword by Sheldon B. Akers. Springer-Verlag, Berlin-New York, 1981, pp. vii+144. ISBN: 3-540-10286-8.