An Almost Classical Logic for Logic Programming and Nonmonotonic Reasoning

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Abstract The model theory of a first-order logic called N^4 is introduced. N^4 does not eliminate double negations, as classical logic does, but instead reduces fourfold negations. N^4 is very close to classical logic: N^4 has two truth values; implications are, in N^4 like in classical logic, material; and negation distributes over compound formulas in N^4 as it does in classical logic. Results suggest that the semantics of normal logic programs is conveniently formalized in N^4: Classical logic Herbrand interpretations generalize straightforwardly to N^4; the classical minimal Herbrand model of a positive logic program coincides with its unique minimal N^4 Herbrand model; the stable models of a normal logic program and its so-called complete minimal N^4 Herbrand models coincide.

1 Introduction

This paper first introduces the (classical style) model theory of a first-order logic called N^4. The salient characteristic of N^4 is that it does not eliminate double negations as classical logic does (in N^4, $\neg\neg F$ is not logically equivalent to F and $\neg\neg\neg F$ is not logically equivalent to $\neg F$), but instead it reduces fourfold negations (in N^4, $\neg\neg\neg\neg F$ is logically equivalent to $\neg\neg F$). The name N^4 stresses that fourfold negations are reduced.

Despite its nonstandard treatment of negation, N^4 is very close to classical logic. Like classical logic, N^4 has two truth values, its implication is material (in N^4, $A \rightarrow B$ is logically equivalent to $\neg A \vee B$), and the truth value of a formula is defined recursively in terms of the truth values of its subformulas. Most logical consequences of classical logic hold also in N^4. In particular, negation distributes over compound formulas in N^4 as it does in classical logic. Also, in N^4 F logically implies $\neg\neg F$ (but the converse does not hold) and three laws of excluded middle hold ($F \vee \neg F$, $\neg F \vee \neg\neg F$, and $\neg\neg F \vee \neg\neg\neg F$ are always true but $F \vee \neg\neg\neg F$ might be false in some so-called incomplete N^4 interpretations). Furthermore, a classical logic model of a set $\mathcal{S}$ of formulas is also a N^4 model of $\mathcal{S}$.

This paper investigates formalizing the semantics of normal logic programs using N^4. A few results suggest that N^4 is convenient for this purpose. Classical logic Herbrand interpretations generalize straightforwardly to N^4 as interpretations characterized by the ground atoms and the doubly negated ground atoms (instead of only the ground atoms) they satisfy. The classical minimal Herbrand model of a positive logic program coincides with its (unique) minimal N^4 Herbrand model. Every normal logic program has (in general many) minimal N^4 Herbrand models. The stable models of a normal logic program [8] coincide with its so-called complete minimal N^4 Herbrand models.

This paper is structured as follows. The next section, Section 2, recalls a few syntax notions and introduces terminology and notations. Section 3 defines the model theory of N^4 and gives a few results on logical consequence in N^4. Section 4 is devoted to N^4 Herbrand interpretations. In Section 5, the minimal N^4 Herbrand models of normal logic programs are defined and investigated. Section 6 discusses the intuitive meaning of N^4. Section 7 addresses perspectives and related work. The proofs are given in Appendix.

2 Syntax, Terminology, and Notations

N^4 syntax is that of classical first-order logic. If $\mathcal{L}$ is a first-order language, its (non-empty) set of constants will be noted $Const_{\mathcal{L}}$, the set of its n-ary ($n \geq 1$) function symbols will be noted $Fun^n_{\mathcal{L}}$, and the set of its n-ary ($n \geq 0$) predicate symbols will be noted $Rel^n_{\mathcal{L}}$. A first-order language is assumed to include the *falsum* $\bot$, the unary connective $\neg$, the binary connectives $\wedge$ $\vee$, and the quantifiers $\forall$ $\exists$.

In the following, a fixed first-order language $\mathcal{L}$ is assumed. The terms, ground terms, Herbrand universe, atoms or atomic formulas, formulas, closed formulas, etc. of $\mathcal{L}$ are defined as usual. Note that the falsum $\bot$ is not an atom. n-fold ($n \geq 0$) negations will be noted $\neg^n$. A formula F is in *prefix negation form* if $F = \neg^n G$ ($n \geq 0$) and no negations occur in G. Two additional connectives, $\rightarrow$ and $\leftrightarrow$, and the *verum* $\top$ are defined as follows as shorthand notations: $(F \rightarrow G) := (\neg F \vee G)$, $(F \leftrightarrow G) := ((\neg F \vee G) \wedge (\neg G \vee F))$, and $\top := \neg\bot$.

The following unusual notion of literal will be used.

Definition 1 (N^4 literal). *A* N^4 literal *is an atom, a negated atom, a doubly negated atom, or a threefold negated atom. A* positive N^4 literal *is an atom or a doubly negated atom. A* negative N^4 literal *is a negated atom or a threefold negated atom.*

A *positive program clause (general program clause,* resp.) in $\mathcal{L}$ is an expression of the form $A \leftarrow B_1, \ldots, B_n$ ($n \geq 0$), where A is an atom of $\mathcal{L}$ and $B_1, \ldots,$ and B_n are atoms (atoms or negated atoms, resp.) of $\mathcal{L}$. A *positive* (*normal or general,* resp.) logic program in $\mathcal{L}$ is a finite set of positive (general, resp.) program clauses in $\mathcal{L}$.

3 N^4 Model Theory

N^4 interpretations resemble that of classical logic. A significant difference is that they assign relations not only to predicate symbols, as classical logic interpretations do, but also to doubly negated predicate symbols.

Definition 2 (N^4 Interpretation). *A* N^4 interpretation $\mathcal{I}$ *of* $\mathcal{L}$ *is a pair* $(D_\mathcal{I}, val_\mathcal{I})$ *such that*

1. *$D_\mathcal{I}$ is a non-empty set, called the* domain *or* universe *of* $\mathcal{I}$.
2. *$val_\mathcal{I}$ is an assignment defined as follows:*
   2.1 $val_\mathcal{I}(c) \in D_\mathcal{I}$, *for* $c \in Const_\mathcal{L}$.
   2.2 $val_\mathcal{I}(f)$ *is a function from* $D_\mathcal{I}^n$ *into* $D_\mathcal{I}$, *for* $f \in Fun_\mathcal{L}^n$ $(n \geq 1)$.
   2.3 $val_\mathcal{I}(p) \in \{\mathbf{true}, \mathbf{false}\}$ *and* $val_\mathcal{I}(\neg^2 p) \in \{\mathbf{true}, \mathbf{false}\}$ *such that if* $val_\mathcal{I}(p) = \mathbf{true}$, *then* $val_\mathcal{I}(\neg^2 p) = \mathbf{true}$, *for* $p \in Rel_\mathcal{L}^0$.
   2.4 $val_\mathcal{I}(p) \subseteq val_\mathcal{I}(\neg^2 p) \subseteq D_\mathcal{I}^n$, *for* $p \in Rel_\mathcal{L}^n$ $(n \geq 1)$.

Thus, in a N^4 interpretation only one of the three truth assignments of Figure 1 are possible for a propositional variable p and the positive N^4 literal $\neg^2 p$. Note that if p is true, then $\neg^2 p$ is also true.

A classical logic interpretation trivially induces a N^4 interpretation: It suffices to assign the same truth value or relation to each doubly negated predicate symbol $\neg^2 p$ as to p. However, not every N^4 interpretation corresponds to a classical logic interpretation. For example, the second line of Figure 1 is not possible in classical logic.

p	$\neg^2 p$
true	**true**
false	**true**
false	**false**

Figure1. Possible valuations of p and $\neg^2 p$ in N^4 interpretations

In N^4, variable and term assignments are defined like in classical logic. Both definitions are recalled, so as to introduce the notations used later.

Definition 3 (Variable Assignment). *Let* $\mathcal{I}$ *be a* N^4 *interpretation of* $\mathcal{L}$ *with domain* $D_\mathcal{I}$. *A* variable assignment $\mathcal{V}$ *with respect to* $\mathcal{I}$ *assigns an element of* $D_\mathcal{I}$ *to each variable of* $\mathcal{L}$. *If* $\mathcal{V}$ *is variable assignment with respect to* $\mathcal{I}$, x *is a variable, and* $d \in D_\mathcal{I}$, *then* $\mathcal{V}[d/x]$ *denotes the following variable assignment with respect to* $\mathcal{I}$:

$$\mathcal{V}[d/x](y) := \begin{cases} d & \text{if } y = x \\ \mathcal{V}(y) & \text{otherwise} \end{cases}$$

Definition 4 (Term Assignment). *Let* $\mathcal{I} = (\mathcal{D}_\mathcal{I}, val_\mathcal{I})$ *be a* N^4 *interpretation of* $\mathcal{L}$ *and* $\mathcal{V}$ *a variable assignment with respect to* $\mathcal{I}$. *The* term assignment $val_{\mathcal{I},\mathcal{V}}$ *with respect to* $\mathcal{I}$ *and* $\mathcal{V}$ *is defined as follows:*

1. $val_{\mathcal{I},\mathcal{V}}(x) = \mathcal{V}(x)$, *for* x *variable.*
2. $val_{\mathcal{I},\mathcal{V}}(c) = val_\mathcal{I}(c)$, *for* $c \in Const_\mathcal{L}$.
3. $val_{\mathcal{I},\mathcal{V}}(f(t_1, \ldots, t_n)) = val_\mathcal{I}(f)(val_{\mathcal{I},\mathcal{V}}(t_1), \ldots, val_{\mathcal{I},\mathcal{V}}(t_n))$, *for* $f \in Fun_\mathcal{L}^n$ $(n \geq 1)$ *and* $t_1, \ldots t_n$ *terms.*

p	$\neg p$	$\neg^2 p$	$\neg^3 p$
true	**false**	**true**	**false**
false	**true**	**true**	**false**
false	**true**	**false**	**true**

Figure2. Possible valuations of p, $\neg p$, $\neg^2 p$, and $\neg^3 p$ in N^4 interpretations

The truth value of a negated formula is defined in such a way, that Figure 1 can be completed as shown by Figure 2, i.e. $\neg p$ negates p but not $\neg^2 p$ and $\neg^3 p$ negates $\neg^2 p$ but not p.

Definition 5 (Formula Valuation). *Let $\mathcal{I} = (\mathcal{D}_\mathcal{I}, val_\mathcal{I})$ be a N^4 interpretation of $\mathcal{L}$ and $\mathcal{V}$ a variable assignment with respect to $\mathcal{I}$. The valuation function $val_{\mathcal{I},\mathcal{V}}$ with respect to $\mathcal{I}$ and $\mathcal{V}$ is defined as follows:*

1.1 $val_{\mathcal{I},\mathcal{V}}(p) = \mathbf{true}$ *iff* $val_\mathcal{I}(p) = \mathbf{true}$, *for* $p \in Rel^0_\mathcal{L}$ *and* $val_{\mathcal{I},\mathcal{V}}(p(t_1,\dots,t_n)) = \mathbf{true}$ *iff* $(val_{\mathcal{I},\mathcal{V}}(t_1),\dots,val_{\mathcal{I},\mathcal{V}}(t_n)) \in val_\mathcal{I}(p)$, *for* $p \in Rel^n_\mathcal{L}$ $(n \geq 1)$ *and* $t_1,\dots,t_n$ *terms.*
1.2 $val_{\mathcal{I},\mathcal{V}}((F_1 \wedge F_2)) = \mathbf{true}$ *iff* $val_{\mathcal{I},\mathcal{V}}(F_1) = \mathbf{true}$ *and* $val_{\mathcal{I},\mathcal{V}}(F_2) = \mathbf{true}$.
1.3 $val_{\mathcal{I},\mathcal{V}}((F_1 \vee F_2)) = \mathbf{true}$ *iff* $val_{\mathcal{I},\mathcal{V}}(F_1) = \mathbf{true}$ *or* $val_{\mathcal{I},\mathcal{V}}(F_2) = \mathbf{true}$.
1.4 $val_{\mathcal{I},\mathcal{V}}(\forall x F) = \mathbf{true}$ *iff for all* $d \in D_\mathcal{I}$ $val_{\mathcal{I},\mathcal{V}[d/x]}(F) = \mathbf{true}$.
1.5 $val_{\mathcal{I},\mathcal{V}}(\exists x F) = \mathbf{true}$ *iff for some* $d \in D_\mathcal{I}$ $val_{\mathcal{I},\mathcal{V}[d/x]}(F) = \mathbf{true}$.
2.1 $val_{\mathcal{I},\mathcal{V}}(\neg\bot) = \mathbf{true}$, $val_{\mathcal{I},\mathcal{V}}(\neg p) = \mathbf{true}$ *iff* $val_{\mathcal{I},\mathcal{V}}(p) \neq \mathbf{true}$, *for* $p \in Rel^n_\mathcal{L}$ $(n \geq 0)$, *and* $val_{\mathcal{I},\mathcal{V}}(\neg p(t_1,\dots,t_n)) = \mathbf{true}$ *iff* $(val_{\mathcal{I},\mathcal{V}}(t_1),\dots,val_{\mathcal{I},\mathcal{V}}(t_n)) \notin val_\mathcal{I}(p)$, *for* $p \in Rel^n_\mathcal{L}$ $(n \geq 1)$.
2.2 $val_{\mathcal{I},\mathcal{V}}(\neg(F_1 \wedge F_2)) = val_{\mathcal{I},\mathcal{V}}((\neg F_1 \vee \neg F_2))$.
2.3 $val_{\mathcal{I},\mathcal{V}}(\neg(F_1 \vee F_2)) = val_{\mathcal{I},\mathcal{V}}((\neg F_1 \wedge \neg F_2))$.
2.4 $val_{\mathcal{I},\mathcal{V}}(\neg\forall x F) = val_{\mathcal{I},\mathcal{V}}(\exists x \neg F)$.
2.5 $val_{\mathcal{I},\mathcal{V}}(\neg\exists x F) = val_{\mathcal{I},\mathcal{V}}(\forall x \neg F)$.
3.1 $val_{\mathcal{I},\mathcal{V}}(\neg^2 p) = \mathbf{true}$ *iff* $val_\mathcal{I}(\neg^2 p) = \mathbf{true}$, *for* $p \in Rel^0_\mathcal{L}$ *and* $val_{\mathcal{I},\mathcal{V}}(\neg^2 p(t_1,\dots,t_n)) = \mathbf{true}$ *iff* $(val_{\mathcal{I},\mathcal{V}}(t_1),\dots,val_{\mathcal{I},\mathcal{V}}(t_n)) \in val_\mathcal{I}(\neg^2 p)$, *for* $p \in Rel^n_\mathcal{L}$ $(n \geq 1)$ *and* $t_1,\dots,t_n$ *terms.*
3.2 $val_{\mathcal{I},\mathcal{V}}(\neg^2(F_1 \wedge F_2)) = val_{\mathcal{I},\mathcal{V}}((\neg^2 F_1 \wedge \neg^2 F_2))$.
3.3 $val_{\mathcal{I},\mathcal{V}}(\neg^2(F_1 \vee F_2)) = val_{\mathcal{I},\mathcal{V}}((\neg^2 F_1 \vee \neg^2 F_2))$.
3.4 $val_{\mathcal{I},\mathcal{V}}(\neg^2\forall x F) = val_{\mathcal{I},\mathcal{V}}(\forall x \neg^2 F)$.
3.5 $val_{\mathcal{I},\mathcal{V}}(\neg^2\exists x F) = val_{\mathcal{I},\mathcal{V}}(\exists x \neg^2 F)$.
4 $val_{\mathcal{I},\mathcal{V}}(\neg^3 F) = \mathbf{true}$ *iff* $val_{\mathcal{I},\mathcal{V}}(\neg^2 F) \neq \mathbf{true}$.
5 $val_{\mathcal{I},\mathcal{V}}(F) = \mathbf{false}$ *iff* $val_{\mathcal{I},\mathcal{V}}(F) \neq \mathbf{true}$.

On can prove as follows that $val_{\mathcal{I},\mathcal{V}}$ is a total function over the formulas of $\mathcal{L}$. For each formula F exactly one of the clauses 1.1 to 4 of Definition 5 apply. (Which clause applies to a formula depends on its structure.) Therefore, $val_{\mathcal{I},\mathcal{V}}(.) = \mathbf{true}$ defines a partial function. Because of clause 5, $val_{\mathcal{I},\mathcal{V}}$ is total. Since $val_{\mathcal{I},\mathcal{V}}$ is a total function, Definition 5 correctly specifies the valuation of formulas in an N^4 interpretation.

Definition 5 differs from its classical logic counterpart as follows. To obtain the definition of classical logic, drop clauses 2.1 through 3.5 and replace clause 4 by:

4' $val_{\mathcal{I},\mathcal{V}}(\neg F) = \mathbf{true}$ *iff* $val_{\mathcal{I},\mathcal{V}}(F) \neq \mathbf{true}$.

Note that, although clauses 2.1 through 3.5 are not needed in the classical logic counterpart of Definition 5, they hold in classical logic. Note also that the elimination of double negation follows from clause 4'.

Properties of N^4 are given in the rest of this section. From now on, $\mathcal{I} = (\mathcal{D}_{\mathcal{I}}, val_{\mathcal{I}})$ denotes a N^4 interpretation of $\mathcal{L}$, $\mathcal{V}$ a variable assignment with respect to $\mathcal{I}$, and F, F_1, F_2, F_3, G, G_1, and G_2 formulas of $\mathcal{L}$.

Proposition 1.

1. $val_{\mathcal{I},\mathcal{V}}(\bot) = \mathbf{false}$ *and* $val_{\mathcal{I},\mathcal{V}}(\top) = \mathbf{true}$
2. $val_{\mathcal{I},\mathcal{V}}((F_1 \wedge F_2)) = val_{\mathcal{I},\mathcal{V}}((F_2 \wedge F_1))$.
3. $val_{\mathcal{I},\mathcal{V}}((F_1 \vee F_2)) = val_{\mathcal{I},\mathcal{V}}((F_2 \vee F_1))$.
4. $val_{\mathcal{I},\mathcal{V}}((F_1 \wedge (F_2 \vee F_3))) = val_{\mathcal{I},\mathcal{V}}(((F_1 \wedge F_2) \vee (F_1 \wedge F_3)))$.
5. $val_{\mathcal{I},\mathcal{V}}((F_1 \vee (F_2 \wedge F_3))) = val_{\mathcal{I},\mathcal{V}}(((F_1 \vee F_2) \wedge (F_1 \vee F_3)))$.
6. $val_{\mathcal{I},\mathcal{V}}((F_1 \wedge (F_2 \wedge F_3))) = val_{\mathcal{I},\mathcal{V}}(((F_1 \wedge F_2) \wedge F_3))$.
7. $val_{\mathcal{I},\mathcal{V}}((F_1 \vee (F_2 \vee F_3))) = val_{\mathcal{I},\mathcal{V}}(((F_1 \vee F_2) \vee F_3))$.
8. *If* $val_{\mathcal{I},\mathcal{V}}(G_1) = val_{\mathcal{I},\mathcal{V}}(G_2)$, *then*
   $val_{\mathcal{I},\mathcal{V}}((F \vee G_1)) = val_{\mathcal{V},\mathcal{I}}((F \vee G_2))$
   $val_{\mathcal{I},\mathcal{V}}((F \wedge G_1)) = val_{\mathcal{V},\mathcal{I}}((F \wedge G_2))$
   $val_{\mathcal{I},\mathcal{V}}(\neg G_1) = val_{\mathcal{V},\mathcal{I}}(\neg G_2)$
9. $val_{\mathcal{I},\mathcal{V}}(\neg\forall x F) = val_{\mathcal{I},\mathcal{V}}(\exists x \neg F)$
10. $val_{\mathcal{I},\mathcal{V}}(\neg\exists x F) = val_{\mathcal{I},\mathcal{V}}(\forall x \neg F)$

Proposition 2. *If F is in prefix negation form and for all $k \in \mathbb{N}$ $F \neq \neg^{2k+1}G$, then $val_{\mathcal{I},\mathcal{V}}(F) \neq val_{\mathcal{I},\mathcal{V}}(\neg F)$.*

As Figures 2 and 3 show, Proposition 2 does not generalize to all formulas.

Proposition 3.

1. Fourfold negation reduction: $val_{\mathcal{I},\mathcal{V}}(\neg^4 F) = val_{\mathcal{I},\mathcal{V}}(\neg^2 F)$
2. Laws of excluded middle:
   $val_{\mathcal{I},\mathcal{V}}((F \vee \neg F)) = val_{\mathcal{I},\mathcal{V}}((\neg F \vee \neg^2 F)) = val_{\mathcal{I},\mathcal{V}}((\neg^2 F \vee \neg^3 F)) = \mathbf{true}$
3. Laws of excluded contradiction:
   $val_{\mathcal{I},\mathcal{V}}((\neg F \wedge \neg^2 F)) = val_{\mathcal{I},\mathcal{V}}((\neg^2 F \wedge \neg^3 F)) = val_{\mathcal{I},\mathcal{V}}((F \wedge \neg^3 F)) = \mathbf{false}$

p	$\neg p$	$\neg^2 p$	$\neg^3 p$
false	**true**	**true**	**false**

Figure3. A N^4 interpretation falsifying $p \vee \neg^3 p$

Although $F \vee \neg F$, $\neg F \vee \neg^2 F$, and $\neg^2 F \vee \neg^3 F$ are true in all N^4 interpretations (Proposition 3), $F \vee \neg^3 F$ might be false in some N^4 interpretations. This is for example the case of $F = p$ in the N^4 interpretation of Figure 3.

Proposition 4. *If F is in prefix negation form, for all $k \in \mathbb{N}$ $F \neq \neg^{2k+1}G$, and $val_{\mathcal{I},\mathcal{V}}(F) = \mathbf{true}$, then $val_{\mathcal{I},\mathcal{V}}(\neg^2 F) = \mathbf{true}$.*

In N^4, implications are defined in terms of negation and disjunction. In contrast to classical logic, in N^4 not all disjunctions are expressible in terms of implications. For some formulas F_1 and F_2, N^4 interpretations $\mathcal{I}$, and variable assignments $\mathcal{V}$, $val_{\mathcal{I},\mathcal{V}}((\neg F_1 \rightarrow F_2)) = \mathbf{true}$ and $val_{\mathcal{I},\mathcal{V}}((F_1 \vee F_2)) = \mathbf{false}$. This is the case, e.g. if F_1 and F_2 are propositional variables and if $\mathcal{I}$ evaluates F_1 and F_2 as shown on Figure 4.

F_1	$\neg^2 F_1$	F_2	$\neg^2 F_2$
false	**true**	**false**	**true**
false	**true**	**false**	**false**

Figure4. A N^4 interpretation s.t. $(\neg F_1 \rightarrow F_2)$ is true and $(F_1 \vee F_2)$ false

Example 1. Let $F_1 = (\neg p \rightarrow p)$. According to Definition 5, in N^4 F_1 is logically equivalent to $(\neg^2 p \vee p)$. Figure 5 gives the possible valuations of p, $\neg p$, $\neg^2 p$ and $\neg^3 p$ in N^4 interpretations in which F_1 and F_2 are true. In classical logic, only the first of these valuations is possible.

p	$\neg p$	$\neg^2 p$	$\neg^3 p$
true	**false**	**true**	**false**
false	**true**	**true**	**false**

Figure5. N^4 interpretations satisfying $(\neg p \rightarrow p)$

Example 2. Let $\mathcal{S}_2 = \{(\neg a \rightarrow b), (\neg b \rightarrow a)\}$. In N^4, $\mathcal{S}_2$ is logically equivalent to $\{(\neg^2 a \vee b), (\neg^2 b \vee a)\}$. Figure 6 gives the two possible valuations of a, $\neg^2 a$, b, and $\neg^2 b$ in N^4 interpretations in which $\mathcal{S}_2$ is true.

Example 3. Let $\mathcal{S}_3 = \{(\neg p \rightarrow p), (p \rightarrow p)\}$. In N^4, $\mathcal{S}_3$ is logically equivalent to $\{(\neg^2 p \vee p), (\neg p \vee p)\}$. Figure 7 gives the possible valuations of p, $\neg p$, and $\neg^2 p$ in N^4 interpretations in which $\mathcal{S}_3$ is true.

While a classical logic interpretation can be seen as N^4 interpretations, some N^4 interpretations have no counterparts in classical logic. Such N^4 interpretations are conveniently characterized as follows.

Definition 6 (In/Complete N^4 Interpretation). *A N^4 interpretation $\mathcal{I}$ of $\mathcal{L}$ is F-*incomplete*, if for some variable assignment $\mathcal{V}$ with respect to $\mathcal{I}$ $val_{\mathcal{I},\mathcal{V}}(F) \neq val_{\mathcal{I},\mathcal{V}}(\neg^2 F)$. Otherwise, it is F-*complete*. A N^4 interpretation is* incomplete*, if it is F-incomplete for some formula F. Otherwise, it is* complete.

a	$\neg^2 a$	b	$\neg^2 b$
true	**true**	**true**	**true**
true	**true**	**false**	**true**
true	**true**	**false**	**false**
false	**true**	**true**	**true**
false	**true**	**false**	**true**
false	**false**	**true**	**true**

Figure6. N^4 interpretations satisfying $\mathcal{S}_2 = \{(\neg a \to b), (\neg b \to a)\}$

p	$\neg p$	$\neg^2 p$
true	**false**	**true**
false	**true**	**true**

Figure7. N^4 interpretation satisfying $\mathcal{S}_3 = \{(\neg p \to p), (p \to p)\}$

Proposition 5. *The following assertions are equivalent:*

1. $\mathcal{I}$ *induces a classical logic interpretation.*
2. $\mathcal{I}$ *is complete.*
3. *For all atoms* A, $\mathcal{I}$ *is* A*-complete.*

Definition 7 (N^4 Model). $\mathcal{I}$ *is a* N^4 model *of* F, *if* $val_{\mathcal{I},\mathcal{V}}(F) = \mathbf{true}$ *for some variable assignment* $\mathcal{V}$ *with respect to* $\mathcal{I}$. *A formula is* N^4 satisfiable *if it has a* N^4 *model. A formula is* N^4 falsifiable, *if there exists a* N^4 *interpretation in which this formula is false.*

"$\mathcal{I}$ is a N^4 model of F" will be noted $\mathcal{I} \models_{\mathrm{N}^4} F$. "$F_2$ logically follows from F_1 in N^4" will be noted $F_1 \models_{\mathrm{N}^4} F_2$.

4 N^4 Herbrand Interpretations

The classical definitions of the Herbrand base and Herbrand interpretations generated by a subset of the Herbrand base extend straightforwardly to N^4.

Definition 8 (N^4 Herbrand Interpretation). *Let* $U_{\mathcal{L}}$ *denote the Herbrand universe of* $\mathcal{L}$, *i.e. the set of all ground terms of* $\mathcal{L}$. *A* N^4 *interpretation* $\mathcal{I} = (D, val)$ *is a* N^4 *Herbrand interpretation if*

1. $D = U_{\mathcal{L}}$
2. *For all* $c \in Const_{\mathcal{L}}$, $val(c) = c$.
3. *For all* $n \in \mathbb{N} \setminus \{0\}$, $f \in Fun^0_{\mathcal{L}}$, *and* $(t_1, \ldots, t_n) \in {U_{\mathcal{L}}}^n, val(f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

Definition 9 (N^4 Herbrand Base). *The* N^4 Herbrand base $B^2_{\mathcal{L}}$ *of* $\mathcal{L}$ *is the set of all positive ground* N^4 *literals of* $\mathcal{L}$. $M \subseteq B^2_{\mathcal{L}}$ *is* closed *if for all atom* $A \in M$, $\neg^2 A \in M$.

Thus, if $B_{\mathcal{L}}$ denotes the classical Herbrand base of a first-order language $\mathcal{L}$, then in $B_{\mathcal{L}} \subseteq B^2_{\mathcal{L}}$ and (if $\mathcal{L}$ has some predicate symbols) $B_{\mathcal{L}} \neq B^2_{\mathcal{L}}$.

Definition 10 **($\mathcal{H}^2(M)$).** *Let M be a closed subset of $B^2_{\mathcal{L}}$. The unique* N^4 *Herbrand interpretation $\mathcal{H}^2(M)$ such that for all positive ground* N^4 *literals $L \in B^2_{\mathcal{L}}$*

$$\mathcal{H}^2(M) \models_{\mathrm{N}^4} L \textit{ iff } L \in M$$

is the N^4 Herbrand interpretation generated by *M.*

The uniqueness of $\mathcal{H}^2(M)$ asserted in Definition 10 follows immediately from Definition 8.

The order on classical interpretations extends to N^4 interpretations.

Definition 11 (Order on N^4 Interpretations). *Let $\mathcal{I}_1 = (D_1, val_1)$ and $\mathcal{I}_2 = (D_2, val_2)$ be two* N^4 *interpretations of $\mathcal{L}$. $\mathcal{I}_1$ is a* sub-interpretation *of $\mathcal{I}_2$, noted $\mathcal{I}_1 \subseteq \mathcal{I}_2$, if*

1. *$D_1 \subseteq D_2$,*
2. *For all $c \in Const_{\mathcal{L}}$ $val_1(c) = val_2(c)$,*
3. *For all $n \in \mathbb{N} \setminus \{0\}$, $f \in Fun^n_{\mathcal{L}}$, and $(d_1, \ldots, d_n) \in {D_1}^n$, $val_1(f)(d_1, \ldots, d_n) = val_2(f)(d_1, \ldots, d_n)$,*
4. *For all $p \in Rel^0_{\mathcal{L}}$, $val_1(p) = val_2(p)$ and $val_1(\neg^2 p) = val_2(\neg^2 p)$.*
5. *For all $n \in \mathbb{N} \setminus \{0\}$ and $p \in Rel^n_{\mathcal{L}}$, $val_1(p) = val_2(p) \cap {D_1}^n$ and $val_1(\neg^2 p) = val_2(\neg^2 p) \cap {D_1}^n$.*

If in addition $D_1 \neq D_2$, then $\mathcal{I}_1$ is a proper *sub-interpretation of $\mathcal{I}_2$.*

Definition 12 (Intersection of N^4 Interpretations). *Let $\{\mathcal{I}_k \mid k \in C\}$ be a collection of* N^4 *interpretations of $\mathcal{L}$ such that*

1. *$\mathcal{I}_k = (D_k, val_k)$.*
2. *$C \neq \emptyset$. Let k_0 be an element of C.*
3. *$D = \bigcap_{k \in C} D_k \neq \emptyset$.*
4. *For all $c \in Const_{\mathcal{L}}$, $k \in C$, $l \in C$, $val_k(c) = val_l(c)$.*
5. *For all $n \in \mathbb{N} \setminus \{0\}$, $f \in Fun^0_{\mathcal{L}}$, $(d_1, \ldots, d_n) \in D$, $k \in C$, $l \in C$, $val_k(f)(d_1, \ldots, d_n) = val_l(f)(d_1, \ldots, d_n)$.*

$\bigcap_{k \in C} \mathcal{I}_k = (D, val)$ is the N^4 *interpretation with universe $D = \bigcap_{k \in C} D_k$ defined by:*

1. *For all $c \in Const_{\mathcal{L}}$, $val(c) = val_{k_0}(c)$.*
2. *For all $n \in \mathbb{N} \setminus \{0\}$, $f \in Fun^0_{\mathcal{L}}$, $(d_1, \ldots, d_n) \in D$, $k \in C$, $l \in C$, $val(f)(d_1, \ldots, d_n) = val_{k_0}(f)(d_1, \ldots, d_n)$.*
3. *For all $p \in Rel^0_{\mathcal{L}}$, $val(p) =$* **false** *if $val_k(p) =$* **false** *for some $k \in C$, $val(p) =$* **true** *otherwise, and*
   $val(\neg^2 p) =$ **false** *if $val_k(\neg^2 p)$* **false** *for some $k \in C$, $val(\neg^2 p) =$* **true** *otherwise.*
4. *For all $n \in \mathbb{N}$, $p \in Rel^n_{\mathcal{L}}$, $val(p) = \bigcap_{k \in C} val_k(p)$ and $val(\neg^2 p) = \bigcap_{k \in C} val_k(\neg^2 p)$.*

Proposition 6. *Let $\{S_k \mid k \in C\}$ be a collection of subsets of the* N^4 *Herbrand base $B^2_{\mathcal{L}}$.*

$$\mathcal{H}^2(\bigcap_{k\in C} S_k) = \bigcap_{k\in C} \mathcal{H}^2(S_k)$$

Let $\mathcal{P}_c(B^2_{\mathcal{L}})$ be the set of closed subsets of $B^2_{\mathcal{L}}$. Since $\mathcal{P}_c(B^2_{\mathcal{L}})$ is obviously closed under intersection and union, $(\mathcal{P}_c(B^2_{\mathcal{L}}), \subseteq)$ is a complete lattice. It follows from Propositions 6 that $(\mathcal{P}_c(B^2_{\mathcal{L}}), \subseteq)$ induces a complete lattice (the order of which is also noted $\subseteq$) over the N^4 Herbrand interpretations of $\mathcal{L}$. Referring to this ordering, the minimal N^4 Herbrand models of a set of formulas which is satisfiable over N^4 Herbrand interpretations are well-defined. Thus, if M is a closed subset of $B^2_{\mathcal{L}}$ and $\mathcal{S}$ is a set of formulas of $\mathcal{L}$, then $\mathcal{H}^2(M)$ is a minimal N^4 Herbrand model of $\mathcal{S}$ iff:

1. $\mathcal{H}^2(M) \models_{\mathrm{N}^4} \mathcal{S}$.
2. For all B closed subset of M such that $B \neq M$, $\mathcal{H}^2(B) \not\models_{\mathrm{N}^4} \mathcal{S}$.

The following characterization of minimal N^4 Herbrand models is used in the next section.

Proposition 7. *Let $\mathcal{S}$ be a set of formulas of $\mathcal{L}$ and M a closed subset of $B^2_{\mathcal{L}}$. Let $\tilde{M} = \{\neg L \mid L \in B^2_{\mathcal{L}} \setminus M\}$. $\mathcal{H}^2_{\mathcal{L}}(M)$ is a minimal* N^4 *Herbrand model of $\mathcal{S}$ iff*

1. *$\mathcal{H}^2_{\mathcal{L}}(M) \models_{\mathrm{N}^4} \mathcal{S}$.*
2. *For all $L \in M$, $\mathcal{S} \cup \tilde{M} \models_{\mathrm{N}^4} L$.*

5 Minimal N^4 Herbrand Models of Normal Logic Programs

Since double negations are not eliminated in N^4, the following interpretation of program clauses as formulas will be used.

Definition 13 (N^4 clausal form). *The* N^4 clause associated with a general program clause *$A \leftarrow B_1, \ldots, B_n$ is the (closed) formula*

$$\forall x_1 \ldots \forall x_k \ (\ldots (A \vee \neg B_1) \vee \ldots \vee \neg B_n)$$

where $x_1, \ldots, x_k$ are the variables occurring in the literals $A, B_1, \ldots,$ and B_n. A N^4 *interpretation $\mathcal{I}$* satisfies *a general program clause C, if $\mathcal{I}$ is a* N^4 *model of the* N^4 *clause associated with C. Otherwise, it* falsifies *it. A interpretation $\mathcal{I}$* satisfies *(or is a* N^4 model*) of a normal logic program, if it satisfies all its program clauses. Otherwise, it* falsifies *it.*

Thus, the N^4 clause associated with the program clause $a(x, y) \leftarrow b(x, z), \neg c(z), d(y)$ is the formula $(\star)$ $\forall x \forall y \forall z\ (((a(x, y) \vee \neg b(x, z)) \vee \neg^2 c(z)) \vee \neg d(y))$. Note that, in $(\star)$, double negations are not eliminated. Note also that $(\star)$ is logically equivalent (in N^4 and in classical logic) to $\forall x \forall y \forall z\ (((b(x, z) \wedge \neg c(z)) \wedge d(y)) \rightarrow a(x, y))$.

Every normal logic program has a N^4 Herbrand model, since $\mathcal{H}(B^2_{\mathcal{L}})$ is a model of every normal logic program. Indeed, $\mathcal{H}(B^2_{\mathcal{L}})$ satisfies every N^4 clause associated with a general program clause, because such a clause contains at least one positive N^4 literal. Note that the classical minimal Herbrand model of a positive logic program corresponds to its (unique) minimal N^4 Herbrand model.

The following examples suggest that complete minimal N^4 Herbrand models might convey a logic program's intuitive meaning. The first two examples are odd, resp. even length recursion cycles through negation.

Example 4. Let $\mathcal{P}_1 = \{p \leftarrow \neg p\}$. In N^4, $\mathcal{P}_1$ is logically equivalent to $\mathcal{S}_1 = \{(\neg^2 p \vee p)\}$. The unique minimal N^4 Herbrand model of $\mathcal{P}_1$ is $\mathcal{H}^2_{\mathcal{L}}(\{\neg^2 p\})$. Figure 8 gives the valuations of p, $\neg p$, and $\neg^2 p$ in this model (compare with Figure 5). Note that $\mathcal{H}^2_{\mathcal{L}}(\{\neg^2 p\})$ is incomplete.

p	$\neg p$	$\neg^2 p$
false	**false**	**true**

Figure8. Minimal N^4 model of $p \leftarrow \neg p$

Example 5. Let $\mathcal{P}_2 = \{a \leftarrow \neg b \ ; \ b \leftarrow \neg a\}$. In N^4 , $\mathcal{P}_2$ is logically equivalent to $\mathcal{S}_2 = \{(\neg^2 b \vee a), (\neg^2 a \vee b)\}$. The minimal N^4 Herbrand models of $\mathcal{P}_2$ are $\mathcal{H}^2_{\mathcal{L}}(\{a, \neg^2 a\})$, $\mathcal{H}^2_{\mathcal{L}}(\{\neg^2 a, \neg^2 b\})$, and $\mathcal{H}^2_{\mathcal{L}}(\{b, \neg^2 b\})$. Figure 9 gives the valuations of the N^4 literals in these models (compare with Figure 6). $\mathcal{H}^2_{\mathcal{L}}(\{\neg^2 a, \neg^2 b\})$ is incomplete.

a	$\neg a$	$\neg^2 a$	b	$\neg b$	$\neg^2 b$
true	**false**	**true**	**false**	**true**	**false**
false	**false**	**true**	**false**	**false**	**true**
false	**true**	**false**	**true**	**false**	**true**

Figure9. Minimal N^4 models of $\mathcal{P} = \{b \leftarrow \neg a \ ; \ a \leftarrow \neg b\}$

Example 6. Let $\mathcal{P}_3 = \mathcal{P}_1 \cup \mathcal{P}_2 = \{p \leftarrow \neg p, \neg a \ ; \ a \leftarrow \neg b \ ; \ b \leftarrow \neg a\}$. The minimal N^4 Herbrand models of $\mathcal{P}_3$ are $\mathcal{H}^2_{\mathcal{L}}(\{a, \neg^2 a\})$, $\mathcal{H}^2_{\mathcal{L}}(\{\neg^2 a, \neg^2 b\})$, and $\mathcal{H}^2_{\mathcal{L}}(\{\neg^2 p, b, \neg^2 b\})$. Compare with the previous examples.

Example 7. Let $\mathcal{P}_4 = \mathcal{P}_1 \cup \{p \leftarrow p\}$. In N^4, $\mathcal{P}_4$ is logically equivalent to $\mathcal{S}_4 = \{(\neg^2 p \vee p), (\neg p \vee p)\}$. It follows from Proposition 3 (2) that $\mathcal{S}_4$ is logically equivalent to $\mathcal{S}_1 = \{(\neg^2 p \vee p)\}$. Thus, $\mathcal{P}_1$ and $\mathcal{P}_4$ have the same minimal N^4 Herbrand model $\mathcal{H}^2_{\mathcal{L}}(\{\neg^2 p\})$.

Proposition 8. *Let $\mathcal{S}$ be a (possibly infinite) set of ground (general) program clauses. If M is a closed subset of $B^2_{\mathcal{L}}$, let* $\mathrm{Simp}_M(\mathcal{S})$ *denote the set of ground general program clauses obtained from $\mathcal{S}$ as follows:*

1. *First delete all clauses whose bodies contain some negative literal $\neg A$ with $A \in M$.*
2. *Second, delete the negative literals from the bodies of the remaining clauses.*

Let A be a ground atom. $\mathcal{S} \cup \tilde{M} \models_{\mathrm{N}^4} A$ iff $\mathrm{Simp}_M(\mathcal{S}) \models A$.

Proposition 8 does not hold in classical logic. Consider for example $\mathcal{P}_1 = \{p \leftarrow \neg p\}$. Assume that p is the only predicate symbol of $\mathcal{L}$ and let $M = \{p, \neg^2 p\}$. In classical logic $\mathcal{P}_1 \cup \tilde{M} = \mathcal{P}_1 \models p$ but $\mathrm{Simp}_M(\mathcal{P}_1) = \emptyset \not\models p$.

Proposition 9. *Let $\mathcal{P}$ be a normal logic program. A* N^4 *Herbrand model of $\mathcal{P}$ is stable iff it is complete and minimal.*

6 Perspectives and Related Work

The approach presented here seems to enjoy many of the strong and weak principles of [3,4]. E.g. "Cut", "Cautious Monotonicity", and the "Principle of Partial Evaluation" result directly from the the classical-style evaluation function (Definition 5), "Relevance" from N^4 treatment of double negations and from model minimality (Proposition 8). This deserves deeper investigations.

The model theory of N^4 presented in this paper needs to be complemented with a proof theory. First investigations indicate that natural deduction and the tableau method well adapt to N^4. A tableau method for N^4 would provide with a basis for defining a fixpoint-like generation of the minimal N^4 Herbrand models of a normal logic program. Also, it would be useful for program development to have at disposal a backward reasoning method able to detect whether, for some instance $G\sigma$ of a goal G, a logic program has a $G\sigma$-incomplete minimal N^4 Herbrand model.

Publications on the semantics of normal logic programs are numerous – cf. the surveys [2,3,4,5,1,6]. For space reasons, these publications cannot be discussed here in detail. Most of them can be roughly classified in ad hoc definitions of models (such as [8]) for (restricted or unrestricted) normal logic programs, forward reasoning methods for computing models, approaches referring to nonstandard logics (often three-valued logics), and approaches based on program transformations. The approach presented here is of the first and third types. Its particularities are that it is based upon a notion of minimal Herbrand models and that it refers to a nonstandard logic rather close to classical logic. Note interesting similarites with the transformation-based approach of [7,11]. Note also that N^4 can be seen as a four-valued logic (the truth values of which can be read "true", "false", "required", and "not required").

Aspects of the work presented here have been inspired from [9,10] as follows. The interpretation of program clauses as N^4 formulas (Definition 13) is reminiscent of their processing in [9]. The characterization of minimal N^4 Herbrand models (Proposition 7) is an adaptation to N^4 of a result given in [10] for classical logic.

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Appendix: Proofs

Proof of Proposition 1:

1. $val_{\mathcal{I},\mathcal{V}}(\bot) = \mathbf{false}$ iff (Def. 5 (5)) $val_{\mathcal{I},\mathcal{V}}(\bot) \neq \mathbf{true}$. This holds, since none of the cases of Def. 5 give rise to derive $val_{\mathcal{I},\mathcal{V}}(\bot) = \mathbf{true}$.
   $val_{\mathcal{I},\mathcal{V}}(\top) = val_{\mathcal{I},\mathcal{V}}(\neg\bot) = \mathbf{true}$ by Def. 5 (2.1).

2-10. Each statement follows from the corresponding property of the meta-language in which Definition 5 is expressed. ∎

Proof of Proposition 2:

The proof is by induction on the structure of F. Let $k \in \mathbb{N}$ and IH(G) denote: "$val_{\mathcal{I},\mathcal{V}}(G) \neq val_{\mathcal{I},\mathcal{V}}(\neg G)$".

Basis cases:

F is an atom or $F = \bot$. IH($\bot$) holds since Def. 5 (2.1) and Prop. 1 (1).

Induction cases:

1. $F = (F_1 \wedge F_2)$. Assume IH(F_1) and IH(F_2) (ind. hyp.). IH(F) follows from Def. 5 (1.2, 2.2), IH(F_1) and IH(F_2).
2. $F = (F_1 \vee F_2)$. Assume IH(F_1) and IH(F_2) (ind. hyp.).IH(F) follows from Def. 5 (1.3, 2.3), IH(F_1) and IH(F_2).
3. $F = \forall x F_1$. Assume IH(F_1) (ind. hyp.). IH(F) follows from Def. 5 (1.4, 2.4) and IH(F_1).
4. $F = \exists x F_1$. Assume IH(F_1) (ind. hyp.). IH(F) follows from Def. 5 (1.5, 2.5) and IH(F_1). ∎

Proof of Proposition 3:

1. $val_{\mathcal{I},\mathcal{V}}(\neg^4 F) = \mathbf{true}$ iff (Def. 5 (4)) $val_{\mathcal{I},\mathcal{V}}(\neg^3 F) \neq \mathbf{true}$ iff (Def. 5 (4)) $val_{\mathcal{I},\mathcal{V}}(\neg^2 F) = \mathbf{true}$.
2. The proof is by induction on the structure of F. Let IH(G) denote: "$val_{\mathcal{I},\mathcal{V}}((G \vee \neg G)) = val_{\mathcal{I},\mathcal{V}}((\neg G \vee \neg^2 G)) = val_{\mathcal{I},\mathcal{V}}((\neg^2 G \vee \neg^3 G)) = \mathbf{true}$."
   Basis cases:
   F is an atom or $F = \bot$.
   $val_{\mathcal{I},\mathcal{V}}((F \vee \neg F)) = \mathbf{true}$ by Def. 5 (1.3, 2.1).
   $val_{\mathcal{I},\mathcal{V}}((\neg F \vee \neg^2 F)) = \mathbf{true}$ by Def. 5 (1.3, 3.1).
   $val_{\mathcal{I},\mathcal{V}}((\neg^2 F \vee \neg^3 F)) = \mathbf{true}$ by Def. 5 (1.3, 4).
   Induction cases:
   1. $F = \neg F_1$. Assume IH(F_1) (ind. hyp.).
      $val_{\mathcal{I},\mathcal{V}}((F \vee \neg F)) = val_{\mathcal{I},\mathcal{V}}((\neg F_1 \vee \neg^2 F_1)) = \mathbf{true}$ (IH(F_1)).
      $val_{\mathcal{I},\mathcal{V}}((\neg F \vee \neg^2 F)) = val_{\mathcal{I},\mathcal{V}}((\neg^2 F_1 \vee \neg^3 F_1)) = \mathbf{true}$ (IH(F_1)).
      By Def. 5 (3.3), Prop. 1 (1), and IH(F_1):
      $val_{\mathcal{I},\mathcal{V}}((\neg^2 F \vee \neg^3 F)) = val_{\mathcal{I},\mathcal{V}}((\neg^3 F_1 \vee \neg^4 F_1)) =$
      $val_{\mathcal{I},\mathcal{V}}(\neg^2(\neg F_1 \vee \neg^2 F_1)) = val_{\mathcal{I},\mathcal{V}}(\neg^2 \top) = \mathbf{true}$.
   2. $F = (F_1 \wedge F_2)$. Assume IH(F_1) and IH(F_2) (ind. hyp.).
      By Def. 5 (3.3), Prop. 1, IH(F_1), and IH(F_2)
      $val_{\mathcal{I},\mathcal{V}}((F \vee \neg F)) = val_{\mathcal{I},\mathcal{V}}(((F_1 \wedge F_2) \vee \neg(F_1 \wedge F_2))) =$
      $val_{\mathcal{I},\mathcal{V}}((((F_1 \vee \neg F_1) \vee F_2) \wedge ((F_2 \vee \neg F_2) \vee F_1))) =$
      $val_{\mathcal{I},\mathcal{V}}(((\top \vee F_2) \wedge (\top \vee F_1)))) = \mathbf{true}$.
      $val_{\mathcal{I},\mathcal{V}}((\neg F \vee \neg^2 F)) = val_{\mathcal{I},\mathcal{V}}((\neg(F_1 \wedge F_2) \vee \neg^2(F_1 \wedge F_2))) =$
      $val_{\mathcal{I},\mathcal{V}}(((\neg F_1 \vee \neg^2 F_1) \wedge (\neg F_2 \vee \neg^2 F_2))) = val_{\mathcal{I},\mathcal{V}}((\top \wedge \top)) = \mathbf{true}$.
      $val_{\mathcal{I},\mathcal{V}}((\neg^2 F \vee \neg^3 F)) = val_{\mathcal{I},\mathcal{V}}((\neg^2(F_1 \wedge F_2) \vee \neg^3(F_1 \wedge F_2))) =$
      $val_{\mathcal{I},\mathcal{V}}(((\neg^2 F_1 \vee \neg^3 F_1) \wedge (\neg^2 F_2 \vee \neg^3 F_2))) = val_{\mathcal{I},\mathcal{V}}((\top \wedge \top)) = \mathbf{true}$.

3. $F = (F_1 \vee F_2)$. The proof is similar to those of the preceding case.
6. $F = \forall x F_1$. Assume IH(F_1) (ind. hyp.). By Def. 5 (3.3), Prop. 1, and IH(F_1):
   $val_{\mathcal{I},\mathcal{V}}((F \vee \neg F)) = val_{\mathcal{I},\mathcal{V}}((\forall x F_1 \vee \neg \forall x F_1)) = val_{\mathcal{I},\mathcal{V}}(\forall x(F_1 \neg F_1)) = val_{\mathcal{I},\mathcal{V}}(\forall x \top) = val_{\mathcal{I},\mathcal{V}}(\top) =$ **true**.
   $val_{\mathcal{I},\mathcal{V}}((\neg F \vee \neg^2 F)) = val_{\mathcal{I},\mathcal{V}}((\neg \forall x F_1 \vee \neg^2 \forall x F_1)) = val_{\mathcal{I},\mathcal{V}}(\forall x(\neg F_1 \vee \neg^2 F_1)) = val_{\mathcal{I},\mathcal{V}}(\forall x \top) = val_{\mathcal{I},\mathcal{V}}(\top) =$ **true**.
   $val_{\mathcal{I},\mathcal{V}}((\neg^2 F \vee \neg^3 F)) = val_{\mathcal{I},\mathcal{V}}((\neg \forall x F_1 \vee \neg^3 \forall x F_1)) = val_{\mathcal{I},\mathcal{V}}(\forall x(\neg^2 F_1 \vee \neg^3 F_1)) = val_{\mathcal{I},\mathcal{V}}(\forall x \top) = val_{\mathcal{I},\mathcal{V}}(\top) =$ **true**.
7. $F = \exists x F_1$. The proof is similar to those of the preceding case. ∎

Proof of Proposition 4:

First note that $val_{\mathcal{I},\mathcal{V}}(\neg^2 \top) =$ **true** since Def. 5 (4, 3.1). Assume $val_{\mathcal{I},\mathcal{V}}(F) =$ **true**. Then, by Prop. 1 (8) $val_{\mathcal{I},\mathcal{V}}(\neg^2 F) = val_{\mathcal{I},\mathcal{V}}(\neg^2 \top)$. By Def. 5 (4, 2.1), $val_{\mathcal{I},\mathcal{V}}(\neg^2 \top) =$ **true**. Hence, $val_{\mathcal{I},\mathcal{V}}(\neg^2 F) =$ **true**. ∎

Proof of Proposition 5:

$1 \leftrightarrow 2$: 2 is a rephrasing of 1.
$2 \rightarrow 3$: By definition of complete and A-complete N^4 interpretations.
$3 \rightarrow 2$: Let $\mathcal{I}$ be a N^4 interpretation which is A-complete for all atoms A. The proof is by induction on the structure of F.

Basis cases:

1. $F = \bot$. By Prop. 1 (1), $val_{\mathcal{I},\mathcal{V}}(\bot) =$ **false** and $val_{\mathcal{I},\mathcal{V}}(\neg\bot) = val_{\mathcal{I},\mathcal{V}}(\top) =$ **true**. Therefore, $\mathcal{I}$ is $\bot$-complete.
2. F is an atom. $\mathcal{I}$ is F-complete, since by hypothesis, it is A-complete for all atoms A.

Induction cases:

1. $F = (F_1 \wedge F_2)$. Assume that $\mathcal{I}$ is F_1-complete and F_2-complete (ind. hyp.). By Def. 5 (1.2, 3.2), $val_{\mathcal{I},\mathcal{V}}(F) = val_{\mathcal{I},\mathcal{V}}(\neg^2 F)$.
2. $F = (F_1 \vee F_2)$. Assume that $\mathcal{I}$ is F_1-complete and F_2-complete (ind. hyp.). By Def. 5 (1.3, 3.3), $val_{\mathcal{I},\mathcal{V}}(F) = val_{\mathcal{I},\mathcal{V}}(\neg^2 F)$.
3. $F = \forall x F_1$. Assume that $\mathcal{I}$ is F_1-complete (ind. hyp.). By Prop. 1 (9, 10), $val_{\mathcal{I},\mathcal{V}}(F) = val_{\mathcal{I},\mathcal{V}}(\neg^2 F)$.
4. $F = \exists x F_1$. Assume that $\mathcal{I}$ is F_1-complete (ind. hyp.). By Prop. 1 (9, 10), $val_{\mathcal{I},\mathcal{V}}(F) = val_{\mathcal{I},\mathcal{V}}(\neg^2 F)$.
5. $F = \neg F_1$. Assume that $\mathcal{I}$ is F_1-complete (ind. hyp.). By Prop. 1 (8) $val_{\mathcal{I},\mathcal{V}}(F) = val_{\mathcal{I},\mathcal{V}}(\neg F_1) = val_{\mathcal{I},\mathcal{V}}(\neg^3 F_1) = val_{\mathcal{I},\mathcal{V}}(\neg^2 F)$. ∎

Proof of Proposition 6:

As its classical logic counterparts, the result follows directly from the definition of the intersection of interpretations (Def. 12). ∎

Proof of Proposition 7:

Necessary condition: Assume that $\mathcal{H}^2(M)$ is a minimal N^4 Herbrand model of $\mathcal{S}$. Thus, 1 holds. If $M = \emptyset$, then 2 holds trivially. Otherwise, let $L \in M$. Let $\tilde{L} = \neg L$. If $\mathcal{S} \cup \tilde{M} \not\models_{\mathrm{N}^4} L$, then $\mathcal{S} \cup \tilde{M} \cup \{\tilde{L}\}$ has a N^4 Herbrand model, hence also a minimal N^4 Herbrand model, say $\mathcal{H}^2(N)$. By definition of $\mathcal{H}^2(N)$, $\mathcal{H}^2(N) \models_{\mathrm{N}^4} \tilde{L}$. Therefore, $\mathcal{H}^2(N) \not\models_{\mathrm{N}^4} L$ (Prop. 2), i.e. $L \notin N$. Since $\mathcal{H}^2(N) \models_{\mathrm{N}^4} \mathcal{S} \cup \tilde{M}$, $N \subseteq M$. Since $L \in M \setminus N$, $N \neq M$. This contradict the minimality of $\mathcal{H}^2(M)$ since $\mathcal{H}^2(N) \models_{\mathrm{N}^4} \mathcal{S}$. Therefore, for all $L \in M$, $\mathcal{S} \cup \tilde{M} \models_{\mathrm{N}^4} L$, i.e. 2 holds.

Sufficient condition: Assume that 1 and 2 hold. If $\mathcal{H}^2(M)$ is not a minimal N^4 Herbrand model of $\mathcal{S}$, then there exists a closed subset N of M such that $N \neq M$ and $\mathcal{H}^2(N)$ is a minimal N^4 Herbrand model of $\mathcal{S}$. Let $\tilde{N} = \{\neg L \mid L \in B^2_{\mathcal{L}} \setminus N\}$. From the necessary condition, it follows that for all $L \in N$, $\mathcal{S} \cup \tilde{N} \models_{\mathrm{N}^4} L$. Since $N \subset M$, $N \neq M$, there exists $L \in M \setminus N$. By hypothesis 2, $\mathcal{S} \cup \tilde{M} \models_{\mathrm{N}^4} L$. Since $N \subset M$, $\tilde{M} \subset \tilde{N}$. Therefore, $\mathcal{S} \cup \tilde{N} \models_{\mathrm{N}^4} L$. But by definition, $L \in M \setminus N$. Therefore, $\tilde{L} \in \tilde{N}$. Thus, both L and $\tilde{L}$ are true in every model of $\mathcal{S} \cup \tilde{N}$, among others in $\mathcal{H}^2(N)$. This contradicts Prop. 2. Therefore, $\mathcal{H}^2(M)$ is a minimal N^4 Herbrand model of $\mathcal{S}$. ∎

Proof of Proposition 8:

By definition of N^4 interpretations (Def. 2), $\neg^2 A \not\models_{\mathrm{N}^4} A$ for all ground atoms A. Therefore, $\mathcal{S} \cup \tilde{M} \models_{\mathrm{N}^4} A$ for some ground atom A iff for all N^4 interpretation $\mathcal{I}$ such that $\mathcal{I} \not\models_{\mathrm{N}^4} M$, there exists a program clause $C \in \mathcal{S}$ such that:

1. A is the head of C.
2. For all positive body literal B of C, $\mathcal{I} \models_{\mathrm{N}^4} B$.
3. For all negative body literals $\neg B$ of C, $\mathcal{I} \not\models_{\mathrm{N}^4} B$.

Thus, $\mathcal{S} \cup \tilde{M} \models_{\mathrm{N}^4} A$ for some ground atom A iff $\mathrm{Simp}_M(\mathcal{S}) \models_{\mathrm{N}^4} A$. Since no negative literal occur in the program clauses in $\mathrm{Simp}_M(\mathcal{S})$, $\mathrm{Simp}_M(\mathcal{S}) \models_{\mathrm{N}^4} A$ implies $\mathrm{Simp}_M(\mathcal{S}) \models A$. ∎

Proof of Proposition 9:

Let Ground($\mathcal{P}$) denote the set of ground instances of the program clauses of a normal logic program $\mathcal{P}$. A stable model of $\mathcal{P}$ [8] is a classical logic Herbrand interpretation $\mathcal{H}_{\mathcal{L}}(M)$ ($M \subseteq B_{\mathcal{L}}$) such that, for all atoms A, $A \in M$ iff $\mathrm{Simp}_M(\mathrm{Ground}(\mathcal{P})) \models A$.

If $M \subseteq B_{\mathcal{L}}$, let $\overline{M} = \{\neg A \mid A \in M\}$ and $\overline{\overline{M}} = \{\neg^2 A \mid A \in M\}$.

Necessary condition: Let $\mathcal{H}_{\mathcal{L}}(M)$ be a stable model of $\mathcal{P}$ (i.e. $M \subseteq B_{\mathcal{L}}$). Since $\mathcal{H}_{\mathcal{L}}(M) \models \mathcal{P}$, $\mathcal{H}^2_{\mathcal{L}}(M \cup \overline{\overline{M}})$ is a complete N^4 model of $\mathcal{P}$. Since $\mathcal{H}^2_{\mathcal{L}}(M \cup \overline{\overline{M}})$ is complete, ($\star$) for all $A \in B_{\mathcal{L}}$, $\mathcal{H}^2_{\mathcal{L}}(M \cup \overline{\overline{M}}) \models_{\mathrm{N}^4} \neg^2 A$ iff $\mathcal{H}^2_{\mathcal{L}}(M \cup \overline{\overline{M}}) \models_{\mathrm{N}^4} A$. Let

$A \in M$. Since $\mathcal{H}_{\mathcal{L}}(M)$ is a stable model of $\mathcal{P}$, $\mathrm{Simp}_M(\mathrm{Ground}(\mathcal{P})) \models A$. Therefore, $\mathrm{Simp}_M(\mathrm{Ground}(\mathcal{P})) \models_{\mathrm{N}^4} A$. It follows from $(\star)$ that for all $L \in M \cup \overline{\overline{M}}$, $\mathrm{Simp}_M(\mathrm{Ground}(\mathcal{P})) \models_{\mathrm{N}^4} L$. I.e. by Prop. 7 $\mathcal{H}^2_{\mathcal{L}}(M)$ is a minimal N^4 model of $\mathcal{P}$.

Sufficient condition: Let $\mathcal{H}^2_{\mathcal{L}}(M)$ be a complete and minimal N^4 model of $\mathcal{P}$. Let Let $N = M \cap B_{\mathcal{L}}$. Since $\mathcal{H}^2_{\mathcal{L}}(M)$ is complete, $M = N \cup \overline{\overline{N}}$. Since $\mathcal{H}^2_{\mathcal{L}}(M)$ is a minimal model of $\mathcal{P}$, by Prop. 7, for all $A \in N$, $\mathcal{P} \cup \tilde{M} \models_{\mathrm{N}^4} A$. Therefore, for all $A \in N$, $\mathrm{Simpl}_M(\mathrm{Ground}(\mathcal{P})) \models_{\mathrm{N}^4} A$. By Prop. 8, for all $A \in B_{\mathcal{L}}$, $\mathrm{Simpl}_M(\mathrm{Ground}(\mathcal{P})) \cup \tilde{M} \models A$. I.e. $\mathcal{H}_{\mathcal{L}}(N)$ is a stable model of $\mathcal{P}$. ∎