A Sequent Calculus for Intuitionistic Default Logic

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Abstract

Current research on non-monotonic reasoning shows growing interest on implementation details, so the need for concrete calculi formalizing non-monotonic logics is clearly recognized. On the other hand, there is also an increased number of works combining intuitionistic logic with various kinds of non-monotonic formalisms. As a case in point, intuitionistic versions of both default and autoepistemic logics have been proposed, and tight connections between intuitionistic logic and logic programs (or deductive databases) using hypothetical inferences have been established.

In this paper, we present a sequent calculus for default reasoning in the style of Bonatti with intuitionistic logic as the underlying logical structure. In contrast to other proposals, Bonatti's technique allows a very simple and intuitive specification of the calculus, making it an ideal tool for implementation purposes.

1 Introduction

Non-monotonic reasoning techniques are an important cornerstone towards formalizing commonsense reasoning and play an important role in the theory of logic programming. After extensive theoretical research on properties of various non-monotonic formalisms, there is growing interest in the problem of implementing such systems. Closely related to this implementation task is the need to have suitably defined calculi for non-monotonic logics in order to perform automated deduction.[1] Among the different deduction methods designed for proof search are (cut-free) sequent calculi and tableau systems. So, most calculi devised for non-monotonic reasoning belong to one of these kinds (e.g., sequent and tableau calculi for circumscriptive non-monotonic formalisms can be found in [21, 22, 23]).

One of the most prominent non-monotonic formalism is Reiter's default logic [26]. Fundamental objects in Reiter's system are extensions of (closed) default theories. Since the construction of an extension involves as an integral part a *consistency check*, a formal system for default logic must somehow take into account that certain formulae are *not provable* (to wit, in order that a default is applied, it is necessary that the negations of its justifications must not be derivable). A very simple and elegant sequent calculus for propositional default logic, formalizing *brave reasoning*, was introduced by Bonatti [4]. The ingenious part in that system is the use of an *axiomatization of invalid formulae*, taking care of the consistency check. Such formal systems for non-theorems, so-called *rejection calculi* or (following Varzi [28]) *complementary calculi*, belong to a less-known branch of modern logic tracing back to Lukasiewicz [18]. Bonatti's system consists of three parts: a classical sequent calculus, a complementary sequent calculus and certain

[1] One should observe that usually non-monotonic logics are *not* defined in terms of strict axiomatizations, rather using "external constraints" (like fixed-point conditions) defining certain classes of formulae. Even the proof-theoretical investigations of Kraus, Lehmann and Magidor [16] are not axiomatizations of non-monotonic inference relations, but a discussion *which properties* such relations should enjoy.

default inference rules. The formal objects of the complementary system are "anti-sequents" of the form $\Gamma \not\vdash \Theta$ stating the non-derivability of Θ from Γ. In a follow-up work, Bonatti also gave a sequent calculus for propositional default logic formalizing *skeptical reasoning* [5].

On the other hand, a tableau-based method for several propositional default systems (including Reiter's) is presented in [2]. In that approach, *extensions* are characterized, i.e., a given set is an extension iff a certain default proof in their system can be found. The main goal of [2] is to give a uniform treatment of the several versions of default logic existing in the literature, and to make it possible to introduce a *default methodology* within each given logical system (such as classical logic, modal logic, temporal logic and so on). In particular, they introduce a default tableau system with intuitionistic logic as the underlying logical system. Although Bonatti's approach [4] has a somewhat different goal than that system, we argue that the method of [2] lacks the simplicity and elegance of [4]. Indeed, one has to constantly switch between different "boxes" and has to check several (meta-)conditions in order to obtain a "proof" in their system. Moreover, even for fairly simple default theories, proofs in that method tend to be rather intricate. In Bonatti's system, however, there is a clear-cut distinction between the different elements constituting a default proof, which makes that calculus very appealing.

The aim of this paper is therefore to give a sequent calculus in the style of Bonatti if the underlying logical system is intuitionistic logic, offering a companion to the intuitionistic default system proposed in [2]. The procedure to do this is actually quite simple: we replace the classical calculus and the complementary calculus by suitable intuitionistic versions.

The paper is organized as follows. In Section 2 we introduce the intuitionistic sequent calculus used in the following, and in Section 3 we present the corresponding intuitionistic complementary system. Section 4 gives a brief introduction of intuitionistic default logic, and Section 5 introduces the intuitionistic default sequent calculus. Section 6 concludes with some general comments.

2 Intuitionistic Logic

Intuitionism originated at the beginning of this century as a result of certain philosophical considerations on the foundations of mathematics. It has a *constructive* attitude towards the acceptance of statements. Roughly speaking, an intuitionist accepts a statement if he can find a proof for it, hence *tertium non datur* is not considered valid. We will not go into a philosophical discussion regarding intuitionism, rather we focus on a *calculus* intended as a formal description capturing intuitionism.

We will use a standard propositional language, where $p, q, \ldots$ denote atomic formulae and $A, B, \ldots$ denote composite formulae. The composite formulae are built from the atomic formulae by the usual formation rules using $\wedge$, $\vee$, $\rightarrow$ as primitive logical connectives. Negation is not considered as a primitive symbol, rather it is defined by $\neg A := (A \rightarrow \bot)$ using the nullary connective $\bot$, called the *absurdity symbol*.

A Hilbert-style axiomatization of intuitionistic logic can be obtained from an axiom system for classical logic by replacing $\neg$-elimination ($\neg\neg A \rightarrow A$) by *weak* $\neg$-elimination ($\neg A \rightarrow (A \rightarrow B)$). The notion of a formula *intuitionistically derivable from* a set S of formulae is defined in the usual way. Semantically, intuitionistic formulae can be described by reflexive and transitive *Kripke models*. A formula is intuitionistically valid iff it is true in every world in every Kripke model. For more details on intuitionistic logic, the reader is referred to standard textbooks on intuitionism (see, e.g., van Dalen's chapter on intuitionistic logic in [6]).

Here, we are interested in a *sequent calculus* as a deduction system for intuitionistic logic. Basically, there are two different kinds of sequent calculi for intuitionistic logic, tracing back to Gentzen and Beth, respectively. Roughly speaking, Gentzen's intuitionistic calculus LJ is a "single-succedent" approach, whereas Beth's version is a "multi-succedent" approach. The version we use is due to Dyckhoff [8] and is along Beth's line.[2]

Definition 1 *An* (intuitionistic) sequent S *is an ordered tuple of the form* $\Gamma \vdash \Sigma$*, where* Γ, Σ *are finite sequences of propositional formulae.* Γ *is the* antecedent *of* S*, and* Σ *is the* succedent *of* S*.*[3] *The informal meaning of a sequent* $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ *is the same as the informal meaning of the formula* $(\bigwedge_{i=1}^{n} A_i) \rightarrow (\bigvee_{i=1}^{m} B_i)$*.*

A problem concerning intuitionistic sequent calculi is that of looping. During proof search, it may happen that formulae have to be considered again and again (from a semantical point of view, this is due to the transitivity property of Kripke models). The problematic rule is the introduction of an implication in the antecedent of a sequent. This rule may have the following form:

$$\frac{(A \rightarrow B), \Gamma \vdash A \quad B, \Gamma \vdash \Sigma}{(A \rightarrow B), \Gamma \vdash \Sigma} \rightarrow l$$

In this rule, a duplicate of $(A \rightarrow B)$ occurs in one of the premises of the rule. Hence, in a bottom-up search, it may not be possible to reduce the logical complexity of a sequent, resulting in a potential non-termination of the search procedure.

Several calculi have been proposed in the literature avoiding the problem of looping [8, 15]. In [8], two loop-free calculi are introduced: LJT and LJT*. LJT is a single-succedent calculus, LJT* is a multi-succedent calculus (the "T" stands for *terminating*). Basically, the idea is to replace the problematic rule $\rightarrow l$ by four rules each representing a certain subcase. Furthermore, no structural rules are present in these calculi; in particular, no contraction rules are used. In the following, we use the calculus LJT*. The axioms and inference rules of LJT* are depicted below.

System LJT*: Axioms

$$A, \Gamma \vdash A, \Sigma \qquad\qquad \bot, \Gamma \vdash \Sigma$$

System LJT*: Logical Rules

$$\frac{A, B, \Gamma \vdash \Sigma}{(A \wedge B), \Gamma \vdash \Sigma} \wedge l \qquad\qquad \frac{\Gamma \vdash A, \Sigma \quad \Gamma \vdash B, \Sigma}{\Gamma \vdash (A \wedge B), \Sigma} \wedge r$$

$$\frac{A, \Gamma \vdash \Sigma \quad B, \Gamma \vdash \Sigma}{(A \vee B), \Gamma \vdash \Sigma} \vee l \qquad\qquad \frac{\Gamma \vdash A, B, \Sigma}{\Gamma \vdash (A \vee B), \Sigma} \vee r$$

[2] For implementations, multi-succedent calculi have certain advantages.

[3] In Gentzen's system LJ, the succedent consists of at most one formula.

$$\frac{B, p, \Gamma \vdash \Sigma}{(p \to B), p, \Gamma \vdash \Sigma} \to l_1 \qquad \frac{(C \to (D \to B)), \Gamma \vdash \Sigma}{((C \land D) \to B), \Gamma \vdash \Sigma} \to l_2$$

$$\frac{(C \to B), (D \to B), \Gamma \vdash \Sigma}{((C \lor D) \to B), \Gamma \vdash \Sigma} \to l_3 \qquad \frac{(D \to B), \Gamma \vdash (C \to D) \quad B, \Gamma \vdash \Sigma}{((C \to D) \to B), \Gamma \vdash \Sigma} \to l_4$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash (A \to B), \Sigma} \to r$$

For the rule $\to l_1$, p is atomic. Observe that Σ does not occur in the premise of the rule $\to r$.

Dyckhoff [8] shows that LJT^* (along with LJT) is equivalent to Gentzen's LJ. Hence, soundness and completeness of LJT^* is established.

Theorem 1 *An intuitionistic sequent $\Gamma \vdash \Sigma$ is provable in LJT^* iff it is intuitionistically valid.*

3 A Rejection Calculus for Intuitionistic Logic

The history of rejected statements traces back as early as Aristotle's theory of syllogisms. Aristotle distinguished between valid and invalid syllogisms (the latter ones are termed "fallacies"). In his attempt to translate Aristotle's system of syllogisms into present-day logic, Łukasiewicz [18] introduced not only a formal system for deducing valid syllogisms, but also a formal system to *reject* invalid ones. So Łukasiewicz was the first modern author who introduced a calculus for invalid statements. However, his system is of a somewhat hybrid nature: it defines rejected statements in terms of valid and invalid ones. More exactly, Łukasiewicz uses the so-called *detachment rule*: If $(A \to B)$ is valid and B is rejected, then A is rejected too.

Łukasiewicz tried also to formalize intuitionistic non-theorems. He conjectured in [19] that the addition of the inference rule "if both A and B are rejected, then so is $(A \lor B)$" to his rejection calculus for propositional logic yields a sound and complete axiomatization for intuitionistic non-theorems. As was shown in [17], this conjecture is false. A correct axiomatization of intuitionistic non-theorems in the style of Łukasziewicz is given in [7]. Similarly, Łukasiewicz-style axiomatizations of certain modal logics can be found in [14, 27].

Preparatory for his default logic calculus, Bonatti introduced in [3] a complementary sequent calculus for classical propositional non-theorems whose description relies purely on the notion of unprovability. Independently, Goranko [14] presented essentially the same calculus as part of his complementary sequent calculi for certain modal logics. As a matter of fact, constructing such a complementary sequent calculus for classical logic is actually quite straightforward due to the fact that the classical sequent inference rules are invertible. However, intuitionistic (and also modal) sequent inference rules are in general not invertible. So the construction of a calculus for invalid statements for those logics is somewhat more complicated. An intuitionistic rejection calculus has been introduced by Pinto and Dyckhoff [24] based on the sequent calculus LJT^*. In the following, we will introduce this calculus, called CRIP (standing for "Calculus for Refutation of Intuitionistic Propositions").

Definition 2 *Let Γ, Θ be finite sequences of propositional formulae. An* anti-sequent *S is an ordered pair of the form $\Gamma \not\vdash \Theta$. As for standard (intuitionistic) sequents, Γ is the antecedent and Θ is the succedent of S.*

Definition 3 *An anti-sequent $\Gamma \not\vdash \Theta$ is* true *iff the corresponding (intuitionistic) sequent $\Gamma \vdash \Theta$ is invalid.*

Definition 4 *An* atomic implication *is a formula of the form $(p \to B)$, where p is some atomic formula.*

The axioms and inference rules of CRIP are given below.

CRIP: Axioms

$$(p_1 \to B_1), \ldots, (p_n \to B_n), \Gamma \not\vdash \Theta$$

For each such axiom, the following proviso applies: Γ and Θ are disjoint and contain only atomic formulae, Θ may additionally contain $\bot$; every p_i is either $\bot$ or an atom not occuring in Γ.[4]

CRIP: Logical Rules

$$\frac{\Gamma, A, B \not\vdash \Theta}{\Gamma, (A \wedge B) \not\vdash \Theta} \wedge l^c$$

$$\frac{\Gamma \not\vdash A, \Theta}{\Gamma \not\vdash (A \wedge B), \Theta} \wedge r_1^c \qquad \frac{\Gamma \not\vdash B, \Theta}{\Gamma \not\vdash (A \wedge B), \Theta} \wedge r_2^c$$

$$\frac{\Gamma, A \not\vdash \Theta}{\Gamma, (A \vee B) \not\vdash \Theta} \vee l_1^c \qquad \frac{\Gamma, B \not\vdash \Theta}{\Gamma, (A \vee B) \not\vdash \Theta} \vee l_2^c$$

$$\frac{\Gamma \not\vdash A, B, \Theta}{\Gamma \not\vdash (A \vee B), \Theta} \vee r^c$$

$$\frac{\Gamma, p, B \not\vdash \Theta}{\Gamma, p, (p \to B) \not\vdash \Theta} \to l_1^c \qquad \frac{\Gamma, (C \to (D \to B)) \not\vdash \Theta}{\Gamma, ((C \wedge D) \to B) \not\vdash \Theta} \to l_2^c$$

$$\frac{\Gamma, (C \to B), (D \to B) \not\vdash \Theta}{\Gamma, ((C \vee D) \to B) \not\vdash \Theta} \to l_3^c \qquad \frac{\Gamma, B \not\vdash \Theta}{\Gamma, ((C \to D) \to B) \not\vdash \Theta} \to l_4^c$$

[4] Recall that $\bot$ is *not* an atom; hence, Γ does not contain $\bot$!

In the next rule, we use the abbreviations $\Gamma' := ((C_1 \rightarrow D_1) \rightarrow B_1), \ldots, ((C_n \rightarrow D_n) \rightarrow B_n), \Gamma$ and $\Gamma_i := \Gamma' - \{((C_i \rightarrow D_i) \rightarrow B_i)\}$.

$$\frac{(D_1 \rightarrow B_1), C_1, \Gamma_1 \not\vdash D_1 \quad \ldots \quad (D_n \rightarrow B_n), C_n, \Gamma_n \not\vdash D_n \qquad \Gamma', E_1 \not\vdash F_1 \quad \ldots \quad \Gamma', E_m \not\vdash F_m}{\Gamma' \not\vdash (E_1 \rightarrow F_1), \ldots, (E_m \rightarrow F_m), \Theta} \rightarrow lr^c$$

The following proviso holds for $\rightarrow lr^c$: Every formula in Γ is either atomic or an atomic implication and none of the antecedents of the atomic implications are equal to any of the atomic formulae. Θ contains only atomic formulae or $\perp$; Γ and Θ are disjoint.

For the rule $\rightarrow lr^c$, some explanations are in order. This rule deviates from the remaining rules of CRIP because it allows the introduction of an implication *on both sides* of an anti-sequent. E.g., for $m = 0, n = 1$ it introduces a nested implication in the antecedent of the conclusion, and for $m = 1, n = 0$ it introduces an implication in the succedent of the conclusion.

The next result paraphrases Theorem 1 of [24].

Theorem 2 *The anti-sequent $\Gamma \not\vdash \Theta$ is provable in* CRIP *iff the intuitionistic sequent $\Gamma \vdash \Theta$ is not provable in* LJT*.*

Corollary 1 *The anti-sequent $\Gamma \not\vdash \Theta$ is provable in* CRIP *iff it is true.*

4 Intuitionistic Default Logic

In this section we describe default logic if the underlying (monotonic) deductive apparatus is intuitionistic logic. Basically, the only difference to usual default logic is that instead of classical logic intuitionistic logic is used. However, certain (intuitionistic) default theories behave differently than their classical counterparts, as we will see below in some examples.

Several non-monotonic formalisms have been proposed using intuitionistic logic. Intuitionistic default logic has been introduced in [2]. To overcome some weaknesses in the early non-monotonic formalisms of McDermott and Doyle [20], Gabbay [11] suggests a reconstruction in terms of intuitionistic logic (later, a fixed point equation was presented in [12]). A conditional intuitionistic logic is considered in [10]. Intuitionistic autoepistemic logic has recently been proposed in [1]. There is a close relationship between intuitionistic logic and logic programs with *hypothetical implications.* Such programs are characterized by allowing rules of the form $(p \leftarrow (q \leftarrow r))$; the assumption of r is a *hypothetical insertion* into the program (or database). As was first observed in [13], logic programs with hypothetical implications can semantically be described in terms of intuitionistic logic.

Definition 5 ([26]) *Let $A, B_1, \ldots, B_n, C$ be propositional formulae. A* default *d is an expression of the form*

$$\frac{A : B_1, \ldots, B_n}{C} .$$

A is the prerequisite*, $B_1, \ldots, B_n$ are the* justifications *and C is the* consequent *of the default d.*

A default theory *is an ordered pair $T = \langle W, D \rangle$, where W is a set of formulae, called the* premises *of T, and D is a set of defaults.*

Intuitively, for a default theory $T = \langle W, D \rangle$, W represents absolute (though in general incomplete) knowledge about the world ("hard facts"), while D represents *defeasible* knowledge ("rules of thumb").

For simplicity, defaults will also be written in the form $(A : B_1, \ldots, B_n / C)$.

In the following, $\mathrm{Th}_I(S)$ denotes the *intuitionistic deductive closure* of a set S of formulae, i.e., $\mathrm{Th}_I(S)$ is the set of all formulae intuitionistically derivable from S.

Definition 6 *Let $T = \langle W, D \rangle$ be a default theory. For any set S of propositional formulae, let $\Gamma_I(S)$ be the* smallest *set K of propositional formulae obeying the following conditions:*

1. *$K = \mathrm{Th}_I(K)$;*
2. *$W \subseteq K$;*
3. *If $(A : B_1, \ldots, B_n / C) \in D$, $A \in K$ and $\neg B_1 \notin S, \ldots, \neg B_n \notin S$, then $C \in K$.*

An extension *of T is a set E of propositional formulae such that $\Gamma_I(E) = E$, i.e., which is a* fixed point *of the operator Γ_I.*

A default theory may have none, one or several extensions. A default theory of the form $\langle W, \emptyset \rangle$ has exactly one extension, namely $\mathrm{Th}_I(W)$. In general, similar to the case using classical logic, appropriate candidates for extensions can be found using the following result.

Proposition 1 *An extension of a default theory $T = \langle W, D \rangle$ is always of the form $\mathrm{Th}_I(W \cup G)$, for some set G such that*

$$G \subseteq \mathrm{CONS}(D) := \{C \mid (A : B_1, \ldots, B_n / C) \in D\}.$$

Example 1 Consider the famous *broken arms* example, given by Poole [25].

Let $T = \langle W, D \rangle$ be the default theory where

$$W := \{broken(left) \vee broken(right)\}$$

and D consists of the defaults

$$\frac{broken(x) \vee \neg broken(x) : usable(x) \wedge \neg broken(x)}{usable(x)}$$

with $x = left, right$.

Let us determine the extensions of T for both classical and intuitionistic logic as the underlying logical structure. First, consider the classical case. Using the classical pendant of Proposition 1, there are only two candidates which may be extensions of T, viz.

$$\begin{aligned} E_1 &:= \mathrm{Th}(W) \\ E_2 &:= \mathrm{Th}(W \cup \{usable(left), usable(right)\}) \end{aligned}$$

where $\mathrm{Th}(\cdot)$ denotes the deductive closure using classical reasoning.

Since in classical logic $broken(x) \vee \neg broken(x)$ is valid and $\neg(usable(x) \wedge \neg broken(x))$ can not be inferred (for $x = left, right$), the two defaults "fire" and $usable(left)$, $usable(right)$ become derivable. Hence, only E_2 is an extension of T. However, the simultaneous adoption of both beliefs $usable(left)$ and $usable(right)$ seems counter-intuitive, because either $broken(left)$ holds or $broken(right)$ holds. On the other hand, with intuitionistic logic, T has the single extension $\mathrm{Th}_I(W)$, due to the fact that $broken(x) \vee \neg broken(x)$ is intuitionistically *not* derivable and hence both defaults are blocked.

This example shows that default logic using classical reasoning can lead to counter-intuitive results, whereas the use of the "finer grained" intuitionistic logic may prevent such unwanted behaviour.

Example 2 Let $T = \langle W, D \rangle$ consist of

$$\begin{array}{lcl} W & := & \{a\}; \\ D & := & \{((\neg\neg p \rightarrow p) : p / \neg p)\}. \end{array}$$

The default in D is potentially "self-destructive", because its application denies part of the reason of applying it in the first place. As a matter of fact, with classical logic, T lacks an extension. Consequently, the initial knowledge a is destroyed by the presence of the self-contradicting default. But with intuitionistic logic, $\mathrm{Th}_I(W)$ is the single extension of T because $(\neg\neg p \rightarrow p)$ is not valid in intuitionistic logic. Hence, in contrast to classical logic, no default is applied and the premise a is preserved.

5 A Calculus for Intuitionistic Default Logic

It is quite interesting to observe that it took more than a decade since Reiter's seminal paper [26] on default logic that someone came up with an appropriate proof theory for it. Although Reiter himself gave a resolution-based characterization of brave reasoning for (first-order) *normal default theories*, i.e., of default theories containing only defaults of the form $(A : B/B)$, strictly speaking this cannot count as a proper proof system because an external (meta-theoretical) consistency check has to be performed.

Bonatti [4] was the first author who introduced a genuine proof-theoretical account for default logic. He developed a sequent calculus for propositional default logic, likewise formalizing brave reasoning as in Reiter's own proposal, but for *general* default theories and not just normal ones. This calculus consists of three parts, namely a propositional sequent calculus, a propositional anti-sequent calculus (taking care of the consistency check for defaults), and certain default inference rules. A calculus for intuitionistic default logic can easily be obtained from Bonatti's system by using LJT^* and CRIP instead of their classical counterparts. We will introduce this calculus now.

Definition 7 *Let Γ, Σ, Θ be finite sequences of propositional formulae, and let Δ be a finite sequence of defaults. A* (brave) default sequent *is an ordered quadrupel of the form* $\Gamma; \Delta \mathrel{|\!\sim} \Sigma; \Theta$.

For a sequence s of arbitrary objects (say, formulae or defaults), $\hat{s}$ is the set of elements of s. Furthermore, for the sake of readability, the empty sequence occurring in a default sequent will be denoted by ϵ.

Definition 8 *A default sequent* $\Gamma;\Delta \mathrel{|\!\!\sim} \Sigma;\Theta$ *is* true *iff there is an extension E of the default theory* $\langle\hat{\Gamma},\hat{\Delta}\rangle$ *such that* $\hat{\Sigma} \subseteq E$ *and* $\hat{\Theta} \cap E = \emptyset$.

The *intuitionistic default sequent calculus* BJ consists of intuitionistic sequents, anti-sequents and default sequents. It incorporates the systems LJT* for intuitionistic sequents and CRIP for anti-sequents. The additional axioms and inference rules of BJ are given below.

System BJ: Axioms

$$\Gamma;\epsilon \mathrel{|\!\!\sim} \epsilon;\epsilon$$

System BJ: Logical Rules

$$\frac{\Gamma \vdash A}{\Gamma;\epsilon \mathrel{|\!\!\sim} A;\epsilon}\; l_1 \qquad\qquad \frac{\Gamma \nvdash A}{\Gamma;\epsilon \mathrel{|\!\!\sim} \epsilon;A}\; l_2$$

$$\frac{\Gamma;\epsilon \mathrel{|\!\!\sim} \Sigma_1;\Theta_1 \quad \Gamma;\epsilon \mathrel{|\!\!\sim} \Sigma_2;\Theta_2}{\Gamma;\epsilon \mathrel{|\!\!\sim} \Sigma_1,\Sigma_2;\Theta_1,\Theta_2}\; cu$$

$$\frac{\Gamma;\Delta \mathrel{|\!\!\sim} \Sigma;\Theta,A}{\Gamma;\Delta,(A:B_1,\ldots,B_n/C) \mathrel{|\!\!\sim} \Sigma;\Theta}\; d_1 \qquad\qquad \frac{\Gamma;\Delta \mathrel{|\!\!\sim} \Sigma,\neg B;\Theta}{\Gamma;\Delta,(A:\ldots,B,\ldots/C) \mathrel{|\!\!\sim} \Sigma;\Theta}\; d_2$$

$$\frac{\Gamma,C;\Delta \mathrel{|\!\!\sim} \Sigma;\Theta,\neg B_1,\ldots,\neg B_n \quad \Gamma;\epsilon \mathrel{|\!\!\sim} A;\epsilon}{\Gamma;\Delta,(A:B_1,\ldots,B_n/C) \mathrel{|\!\!\sim} \Sigma;\Theta}\; d_3$$

It is quite straightforward to check that the proofs of the soundness and completeness results given in [4] still hold good for the intuitionistic case. Therefore, we can state:

Theorem 3 *The default sequent* $\Gamma;\Delta \mathrel{|\!\!\sim} \Sigma;\Theta$ *is derivable in* BJ *iff it is true.*

Example 3 Consider the default theory T of Example 2. We give a proof in the system BJ of the default sequent $a;((\neg\neg p \to p):p/\neg p) \mathrel{|\!\!\sim} a;\neg p$. Recall that $\neg p$ is an abbreviation of $(p \to \bot)$, hence $\neg\neg p$ stands for $((p \to \bot) \to \bot)$.

$$\dfrac{\dfrac{\dfrac{a \vdash a}{a;\epsilon \mathrel{|\!\!\sim} a;\epsilon}\; l_1 \qquad \dfrac{\dfrac{\dfrac{(\bot \to \bot),p,a \nvdash \bot}{a,\neg\neg p \nvdash p}\; \to lr^c\ (*)}{a \nvdash (\neg\neg p \to p)}\; \to lr^c\ (**)}{a;\epsilon \mathrel{|\!\!\sim} \epsilon;(\neg\neg p \to p)}\; l_2}{a;\epsilon \mathrel{|\!\!\sim} a;(\neg\neg p \to p)}\; cu \qquad \dfrac{\dfrac{a,p \nvdash \bot}{a \nvdash \neg p}\; \to lr^c\ (***)}{a;\epsilon \mathrel{|\!\!\sim} \epsilon;\neg p}\; l_2}{\dfrac{a;\epsilon \mathrel{|\!\!\sim} a;(\neg\neg p \to p),\neg p}{a;((\neg\neg p \to p):p/\neg p) \mathrel{|\!\!\sim} a;\neg p}\; d_1}\; cu$$

In $(*)$ and $(***)$, rule $\to lr^c$ is applied for $n=1, m=0$; and in $(**)$, rule $\to lr^c$ is applied for $n=0, m=1$.

6 Conclusion

We presented a sequent calculus for default logic in the style of Bonatti [4] with intuitionistic logic as its underlying (monotonic) inference apparatus, offering a companion to the intuitionistic default tableau system introduced in [2]. The advantage of Bonatti's approach is its simple and intuitive arrangement, making the structure behind the default mechanism clearly visible. This feature is a strong asset if one wants to study proof-theoretical properties of this calculus, e.g., in [9] we showed for a (classical) first-order version of this calculus that it enables non-elementary speed-ups of proof lengths as compared to (classical) monotonic reasoning.

References

[1] L. Aiello, G. Amati and F. Pirri. Intuitionistic Autoepistemic Logic. *Studia Logica*, 59, 1997.

[2] G. Amati, L. C. Aiello, D. Gabbay and F. Pirri. A Proof Theoretical Approach to Default Reasoning I: Tableaux for Default Logic. *Journal of Logic and Computation*, 6(2):205–231, 1996.

[3] P. A. Bonatti. *A Gentzen System for Non-Theorems.* Technical Report CD-TR 93/52, Christian Doppler Labor für Expertensysteme, Technische Universität Wien, Paniglgasse 16, A–1040 Wien, 1993.

[4] P. A. Bonatti. Sequent Calculi for Default and Autoepistemic Logics. *Proceedings TABLEAUX'96*, Springer LNAI 1071, pp. 127–142, 1996.

[5] P. A. Bonatti. A Sequent Calculus for Skeptical Default Logic. *Proceedings TABLEAUX'97*, Springer LNAI 1227, pp. 107–121, 1997.

[6] D. van Dalen. Intuitionistic Logic. *Handbook of Philosophical Logic*, Vol. III, pp. 225–339, 1986.

[7] R. Dutkiewicz. The Method of Axiomatic Rejection for the Intuitionistic Propositional Logic. *Studia Logica*, 48(4):449–460, 1989.

[8] R. Dyckhoff. Contraction-Free Sequent Calculi for Intuitionistic Logic. *The Journal of Symbolic Logic*, 57(3):795–807, 1992.

[9] U. Egly and H. Tompits. Non-Elementary Speed-Ups in Default Reasoning. *Proceedings ECSQARU-FAPR'97*, Springer LNAI 1244, pp. 237–251, 1997.

[10] G. Fischer Servi. Nonmonotonic Consequence Based on Intuitionistic Logic. *Journal of Symbolic Logic*, 57:1167–1197, 1992.

[11] D. Gabbay. Intuitionistic Basis for Non-Monotonic Logic. *Proceedings of the Sixth Conference on Automated Deduction*, Springer LNCS 138, pp. 260–271, 1982.

[12] D. Gabbay and M. R. B. Clarke. An Intuitionistic Basis for Non-Monotonic Reasoning. *Non-Standard Logics for Automated Reasoning*, Academic Press, pp. 163–178, 1988.

[13] D. Gabbay and U. Reyle. N-Prolog: An Extension of Prolog with Hypothetical Implications. *Journal of Logic Programming*, 1(4):319–355, 1984.

[14] V. Goranko. Refutation Systems in Modal Logic. *Studia Logica*, 53(2):299–324, 1994.

[15] J. Hudelmaier. An $O(n \log n)$-Space Decision Procedure for Intuitionistic Propositional Logic. *Journal of Logic and Computation*, 3(1):63–75, 1993.

[16] S. Kraus, D. Lehmann and M. Magidor. Nonmonotonic Reasoning, Preferential Models and Cumulative Logics. *Artificial Intelligence*, 44(1):167–207, 1990.

[17] G. Kreisel and H. Putnam. Eine Unableitbarkeitsbeweismethode für den Intuitionistischen Aussagenkalkül. *Archiv für Mathematische Logik und Grundlagenforschung*, 3:74–78, 1957.

[18] J. Łukasiewicz. *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic.* Clarendon Press, Oxford, 1951.

[19] J. Łukasiewicz. On the Intuitionistic Theory of Deduction. *Indagationes Mathematicae*, 3:202–212, 1952.

[20] D. McDermott and J. Doyle. Non-Monotonic Logic I. *Artificial Intelligence Journal*, 13:41–72, 1980.

[21] I. Niemelä. A Tableau Calculus for Minimal Model Reasoning. *Proceedings TABLEAUX'96*, Springer LNAI 1071, pp. 278–294, 1996.

[22] I. Niemelä. Implementing Circumscription Using a Tableau Method. *Proceedings ECAI'96*, John Wiley & Sons Ltd., Chichester, pp. 80–84, 1996.

[23] N. Olivetti. Tableaux and Sequent Calculus for Minimal Entailment. *Journal of Automated Reasoning*, 9:99–139, 1992.

[24] L. Pinto and R. Dyckhoff. Loop-Free Construction of Counter-Models for Intuitionistic Propositional Logic. *Symposia Gaussiana*, Walter de Gruyter & Co, Berlin, pp. 225–232, 1995.

[25] D. Poole. What the Lottery Paradox Tells Us about Default Reasoning. *Proceedings KR'89*, pp. 333–340, 1989.

[26] R. Reiter. A Logic for Default Reasoning. *Artificial Intelligence Journal*, 13, p. 81, 1980.

[27] T. Skura. Refutations and Proofs in S4. In H. Wansing (ed.), *Proof Theory of Modal Logic*, Kluwer, pp. 45–51, 1996.

[28] A. Varzi. Complementary Sentential Logics. *Bulletin of the Section of Logic*, 19:112–116, 1990.