# Relations between Fuzzy Time Intervals

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**Abstract**

Time intervals like 'tonight', which are usually not very precise, can be modeled as fuzzy sets. But this causes the problem that the relations between points and intervals and between two intervals, which are usually very trivial, become very complex when the intervals are fuzzy sets. Moreover, there are many different possibilities to define such relations. In this paper I propose a very flexibe operator-based approach to point–interval and interval-interval relations, where the intervals are fuzzy time intervals over the real numbers. The relations yield non-trivial fuzzy values even if the intervals are crisp.

Track 1, full paper, keywords: ontologies of time, fuzzy temporal intervals

## 1 Motivation and Introduction

As a motivating example, consider a database with, say, a cinema timetable. If you query the timetable "give me all performances ending *before* midnight", do you really want to exclude a performance ending just one minute after midnight? I think, not. One could solve this problem by giving the 'before' relation a fuzzy meaning, such that performances ending before midnight get a fuzzy value 1, and performances ending after midnight get a fuzzy value which decreases the later the performance ends. The fuzzy value could then be used to order the answers to the query such that the performances ending after midnight come late in the list.

Nagypál and Motik [4] have defined an extension of the interval relations to fuzzy intervals. There are, however, many different ways to define such extensions. In this paper I propose a three-level approach to fuzzy relations between fuzzy intervals. The first level consists of basic operators which can manipulate fuzzy intervals in certain ways. The second level consists of point–interval relations *before*, *starts*, *during*, *finishes*, *after* and a few other ones. These relations are also defined as operators which transform the intervals by using combinations of the basic operators. Finally, at the third level we define interval–interval relations, basically as integrals over the corresponding point–interval relations.

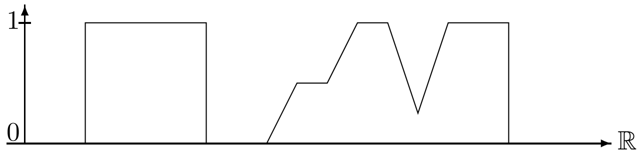
The operators are implemented in the FuTIRe library [5]. This is a component of the WebCal system [2], a program for evaluating temporal expressions like ‘three weeks after Easter’. A prototype of the WebCal system is currently being tested.

# 2 Fuzzy Time Intervals

Fuzzy Intervals are usually defined through their membership functions [6, 3]. A membership function maps a base set to a real number between 0 and 1. The base set for fuzzy time intervals is a linear time axis, isomorphic to the real numbers.

**Definition 2.1 (Fuzzy Time Intervals)** *A* fuzzy membership function *is a total function* $f : \mathbb{R} \mapsto [0, 1]$ *which does not need to be continuous, but it must be integratable. The* fuzzy interval $i_f$ *that corresponds to a fuzzy membership function* $f$ *is* $i_f \stackrel{\text{def}}{=} \{(x, y) \subseteq \mathbb{R} \times [0, 1] \mid y \leq f(x)\}$*. Given a fuzzy interval* $i$ *we usually write* $i(x)$ *to indicate the corresponding membership function.* ■

This definition comprises single or multiple crisp or fuzzy intervals like these:




*Crisp and Fuzzy Intervals*

The fuzzy intervals can also be infinite. For example the term ‘after tonight’ may be represented by a fuzzy value which rises from fuzzy value 0 at 6 pm until fuzzy value 1 at 8 pm and then remains 1 ad infinitum.

Fuzzy time intervals may be quite complex structures with many different characteristic features. The simplest ones are *core* and *support*. The core $C(i)$ is the part of the interval $i$ where the fuzzy value is 1, and the support $S(i)$ is the subset of $\mathbb{R}$ where the fuzzy value of $i$ is non-zero. In addition we define the *kernel* $K(i)$ as the part of the interval $i$ where the fuzzy value is *not* constant ad infinitum, i.e. the kernel is the smallest subset of $\mathbb{R}$ such that $i(x)$ is constant for $x$ outside $K(i)$. Fuzzy time intervals with finite kernel are of particular interest because although they may be infinite, they can easily be implemented with finite datastructures.

Fuzzy time intervals can be measured in various ways. Besides the size $|i| \stackrel{\text{def}}{=} \int i(x)\ dx$, one can locate the position of the core, support and kernel. $i^{fO}$ and $i^{lO}$ are the first and last $x$-coordinates of $O$ where $O$ is either $C$ (core) or $S$ (support) or $K$ (kernel).

One can also measure the maximal fuzzy value $i\hat{}$. This should, but need not be 1. Let $i^{fm}$ be the $x$-coordinate of the first point with $i(i^{fm}) = i\hat{}$ and let $i^{lm}$ be the last such point.

Fuzzy time intervals can consist of several different components. A component is a sub-interval of a fuzzy interval such that the left and right end is either infinity, or the membership function drops down to 0. Let $Cmp(i)$ be the list of components of $i$.

For ordinary intervals there are the standard Boolean set operators: complement, intersection, union etc. These are uniquely defined. There is no choice. There are no such uniquely defined set operators for fuzzy intervals. Set operators are essentially transformations of the membership functions, and there are lots of different ones.

The *complement* $N(i)$ of a fuzzy interval $i$ is usually defined by a *negation function* $n$ such that $N(i)(x) = n(i(x))$. For the *standard complement* one uses the negation function $n(y) = 1 - y$. Another alternative is the $\lambda$-*complement*: $N_\lambda(i)(x) \stackrel{\text{def}}{=} (1 - i(x))/(1 + \lambda \cdot i(x))$.

Intersection and union of two fuzzy intervals are defined by means of *triangular norms* and *conorms*. These are two-place functions on fuzzy values which satisfy certain forms of commutativity, associativity and monotonicity. If $T$ is a triangular norm then $(i \cap_T j)(x) \stackrel{\text{def}}{=} T(i(x), j(x))$ is the definition of intersection and if $S$ is a triangular conorm then $(i \cup_S j)(x) \stackrel{\text{def}}{=} S(i(x), j(x))$ is the definition of union of two fuzzy intervals. The triangular norm for *standard intersection* is $T = \min$ and the triangular conorm for *standard union* is $S = \max$. A particular class of triangular norms and conorms, together with the $\lambda$-complement, is the *Hamacher family*.

$$\begin{array}{lcll} T_\gamma(x,y) & \stackrel{\text{def}}{=} & xy/(\gamma + (1-\gamma)(x+y-xy)) & \gamma \geq 0 \\ S_\beta(x,y) & \stackrel{\text{def}}{=} & (x+y+(\beta-1)xy)/(1+\beta xy) & \beta \geq -1 \end{array}$$

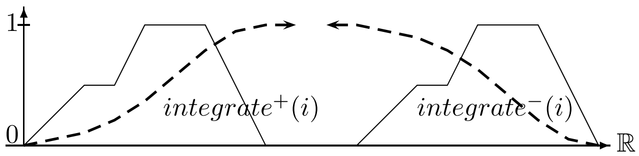
## 3 Basic Transformations

The point–interval relations proposed in this paper can be parameterized with certain interval operators. Therefore we first introduce a part of a little library of interval operators. More of them can be found in [5].

**Definition 3.1 (Basic Transformations)** *Let $i \in F_{\mathbb{R}}$ be a fuzzy interval. We define the following (parameterized) interval operators:*

$$
\begin{array}{lcl}
identity(i) & \stackrel{\text{def}}{=} & i \\
extend^{+}(i)(x) & \stackrel{\text{def}}{=} & i(x) \textit{ if } x \leq i^{fm}, \textit{ and } 1 \textit{ otherwise} \\
extend^{-}(i)(x) & \stackrel{\text{def}}{=} & i(x) \textit{ if } x \geq i^{lm}, \textit{ and } 1 \textit{ otherwise} \\
scaleup(i)(x) & \stackrel{\text{def}}{=} & i(x)/\hat{i} \textit{ if } \hat{i} \neq 0, \textit{ and } 0 \textit{ otherwise} \\
integrate^{+}(i)(x) & \stackrel{\text{def}}{=} & \int_{-\infty}^{x} i(y)dy/|i| \\
integrate^{-}(i)(x) & \stackrel{\text{def}}{=} & \int_{x}^{+\infty} i(y)dy/|i| \\
shift_n(i)(x) & \stackrel{\text{def}}{=} & i(x-n) \\
cut_{x_1,x_2}(i)(x) & \stackrel{\text{def}}{=} & 0 \textit{ if } x < x_1 \textit{ or } x \geq x_2, \textit{otherwise } i(x)
\end{array}
$$

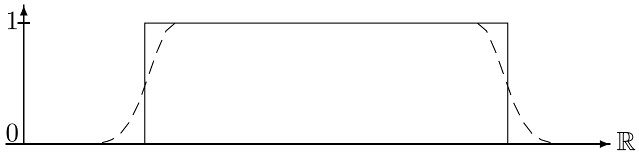
$extend^{+}(i)$ follows the left part of the interval until the left maximum $i^{fm}$ is reached and then stays at fuzzy value 1. $extend^{-}(i)$ is the symmetric version of $extend^{+}(i)$. The *scaleup*-function scales the membership function up such that $scaleup(i)\hat{} = 1$. The *integrate* operators integrate over the membership function and normalize the integral to values $\leq$ 1.




*integrate*$^{+}$ *and integrate*$^{-}$

*extend*$^{+}$ and *integrate*$^{+}$ are *rising operators* and *extend*$^{-}$ and *integrate*$^{-}$ are *falling operators.*

The FuTIRe library has in addition some *fuzzification operators*. They can make crisp (and also fuzzy) intervals fuzzy by multiplying the left or right end of the interval with a linear or Gaussian shape increase or decrease. The next picture illustrates the operation. The mathematical details can be found in [5].




*Gaussian Fuzzification*

## 4 Point–Interval Relations

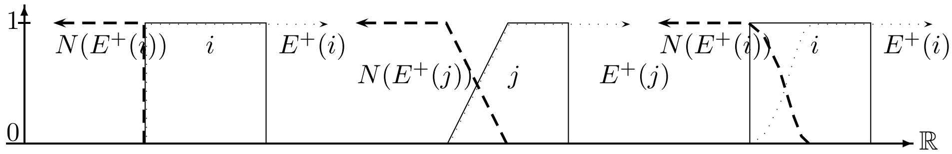
There are five basic relations between a time point and a *crisp* interval $i$: *before, starts, during, finishes*, and *after*. We define these relations as transformation operators. To understand this, consider a point $p$ and an interval $i$. As the result of the relation $p$ *before* $i$ we want a fuzzy value. To get this fuzzy value, we can turn $i$ into another interval *before*$(i)$ such that *before*$(i)(x)$ is the desired fuzzy value. The general definitions are presented first, and then explained in detail.

**Definition 4.1 (Point–Interval Relations as Transformations)** *Let $i$ be a fuzzy interval, and let $T$ be a triangular norm. We define the following point–interval relation operators:*

$\text{before}_{N,E^+}(i) \stackrel{\text{def}}{=} \emptyset$ *if* $i = \emptyset$ *and* $N(E^+(i))$ *otherwise*
*where $N$ is a complement-operator and $E^+$ a rising operator,*
$\text{after}_{N,E^-}(i) \stackrel{\text{def}}{=} \emptyset$ *if* $i = \emptyset$ *and* $N(E^-(i))$ *otherwise*
*where $N$ is a complement-operator and $E^-$ a falling operator,*
$\text{starts}_{E^+,B,T}(i) \stackrel{\text{def}}{=} \mathit{scaleup}(E^+(i) \cap_T B(i))$
*where $E^+$ is a rising operator and $B$ a* before*-operator,*
$\text{finishes}_{E^-,A,T}(i) \stackrel{\text{def}}{=} \mathit{scaleup}(E^-(i) \cap_T A(i))$
*where $E^-$ is a falling operator and $A$ an* after*-operator,*
$\text{during}_{U,O}(i) \stackrel{\text{def}}{=} U_{l \in Cmp(i)} O(l)$
*where $U$ is a union operator and $O$ any unary transformation.*

This list is extended in [5] by relations 'during the $n^{th}$ $m^{th}$ part' (e.g. during the second quarter', 'in the middle of the $n^{th}$ $m^{th}$ part', 'in the gap' and 'in the $k^{th}$ gap'.

**Before** and **After**: In the expression $N(E^+(i))$ for *before*, $E^+$ projects the rising part of $i$ out and hides all the rest of $i$. $N$ complements the rising part. The first two examples in the picture below use $E^+ = \mathit{extend}^+$. The last example uses $E^+ = F \circ \mathit{extend}^+$ where $F$ is a Gaussian fuzzification operator. $N$ is the standard complement in both cases.

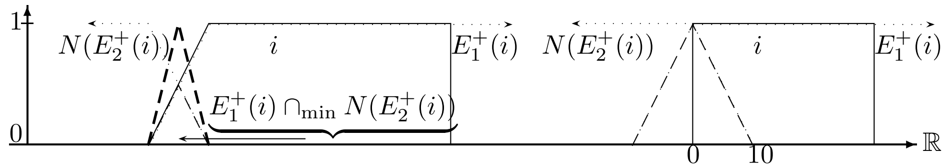



before *with* $N$ = *Standard Negation,* $E^+ = \mathit{extend}^+$ *and* $E^+ = F \circ \mathit{extend}^+$

The *after*-relation is just the symmetric variant of the *before*-relation.

**Starts and Finishes**: The definition of *starts* is $scaleup(E_1^+(i) \cap_T N(E_2^+(i)))$. The rising function $E_1^+$ extracts the front part of $i$. $N(E_2^+(i))$ extracts the same or another front part of $i$ and complements it with the complement function $N$. The first extracted front part and the complemented extracted front part are then intersected. Since the intersection may yield a fuzzy set with maximum fuzzy value $< 1$ it is scaled up to 1.

Using $extend^+$ in combination with a fuzzification operator yields a fuzzy *starts*-relation even for crisp intervals.




*Simple* starts-*Relation* *Fuzzy* starts *for a Crisp Interval*

The dashed line indicates the scaled up intersection. The *finishes* relation is the symmetric variant of the *starts*-relation.

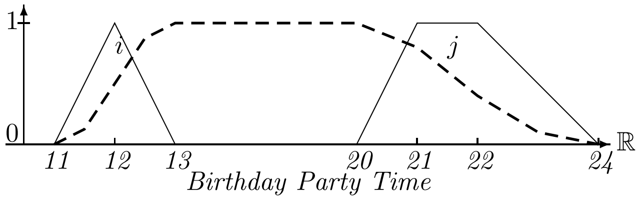
**During**: The simplest version of the *during* operator is just the identity. This returns 0 for all $x$ outside $i$ and the fuzzy membership value for all $x$ inside $i$. We can weaken this strict interpretation by fuzzifying the components of the interval $i$ and taking the membership function of the union of the fuzzified components.

**Until**: With the basic transformations (Def. 3.1) one can define an 'Until' operator between fuzzy intervals. With this operator one can model expressions like 'from early morning until late night', where 'early morning' and 'late night' are concrete fuzzy intervals. The four possible definitions are:

$$\begin{aligned}
Until^{bb}_{E^+,T,N}(i,j) &\stackrel{\text{def}}{=} E^+(i) \cap_T N(E^+(j)) \\
Until^{be}_{E^+,E^-,T}(i,j) &\stackrel{\text{def}}{=} E^+(i) \cap_T E^-(j) \\
Until^{eb}_{E^+,E^-,T,N}(i,j) &\stackrel{\text{def}}{=} N(E^-(i)) \cap_T N(E^+(j)) \\
Until^{ee}_{E^-,T,N}(i,j) &\stackrel{\text{def}}{=} N(E^-(i)) \cap_T E^-(j).
\end{aligned}$$

where $T$ is a t-norm, $N$ a complement operator, $E^+$ a 'rising' transformation (like $extend^+$) and $E^+$ a 'falling' transformation. $Until^{bb}(i,j)$ means 'from the beginning of $i$ until the beginning of $j$'. The other three versions cover the other three cases.

**Example 4.2 (Birthday Party Time)** *We want to model that the birthday party took place 'from around noon until early evening'. Suppose, we have a formalization of 'around noon' as the fuzzy set i in the picture below* *and 'early evening' as the set $j$. The fuzzy value of the birthday party duration at a time point $x$ is 1 if the probability that the party started before $x$ is 1 and the probability that the party ended after $x$ is also 1. Therefore the fuzzy value at point $x$ is computed by integrating over the probabilities of the start points and the end points. To get this behaviour we choose $Until_{be}$ with $E^+ = integrate^+$ and $E^- = integrate^-$ (Def. 3.1). The resulting fuzzy set is:*




## 5 Interval–Interval Relations

Allen's seven interval relations *before, meets, overlaps, starts, during, finishes,* and *equals*, are the basic relations between two crisp intervals [1]. The basic idea for generalizing these relations is very simple: since we have the point–interval relations, we can extend the point to an interval in the relation by integrating over the interval's membership function. We summerize the interval–interval relations in the definition below and then explain them in detail one by one.

**Definition 5.1 (Interval–Interval Relations)**
*Let $i$ and $j$ be two fuzzy time intervals. We need the following point–interval operators (Def. 4.1): $B$ is a Point–Interval* before-*operator, $E^+$ is a rising transformation (like $extend^+$), $D$ is an Interval–Interval* during *operation (defined below), $D_p$ is a point–interval* during-*operator, $S, S_1$ and $S_2$ are point–interval* starts-*operators, $F, F_1$ and $F_2$ are point–interval* finishes-*operators.*

$$\text{before}_B(i,j) \stackrel{\text{def}}{=} \begin{cases} 0 & \textit{if } i \textit{ is empty or positive infinite or } j \textit{ is empty} \\ 1 & \textit{if } i \textit{ is negative infinite and } i \cap_{min} j = \emptyset \\ \int (i \cap_{min} j)(x) \cdot B(j)(x)\ dx/|i \cap_{min} j| & \textit{if } i \textit{ is negative infinite} \\ \int i(x) \cdot B(j)(x)\ dx/|i| & \textit{otherwise} \end{cases}$$

$$\text{meets}_{F,S}(i,j) \stackrel{\text{def}}{=} \begin{cases} 0 & \textit{if } i = \emptyset \textit{ or } j = \emptyset \textit{ or } i \textit{ is pos. infinite or } j \textit{ is neg. infinite} \\ \int F(i)(x) \cdot S(j)(x)\ dx/N(F(i), S(j)) & \textit{otherwise} \end{cases}$$

$$\text{during}_{D_p}(i,j) \stackrel{\text{def}}{=}$$
$$\begin{cases} 1 & \textit{if } i \textit{ is empty} \\ 0 & \textit{if } j \textit{ is empty or } i \textit{ is infinite and } i(-\infty) > j(-\infty) \textit{ or } i(+\infty) > j(+\infty) \\ \int i'(x), D_p(j')(x)\ dx/|i'| & \textit{if } i \textit{ is infinite} \\ \int i(x) \cdot D_p(j)(x)\ dx/|i| & \textit{otherwise} \end{cases}$$
*where* $i' = cut_{min(i^{fK},j^{fK}),max(i^{lK},j^{lK})}(i)$ *and* $j' = cut_{min(i^{fK},j^{fK}),max(i^{lK},j^{lK})}(j)$

$$\text{overlaps}_{E+,D}(i,j) \stackrel{\text{def}}{=}$$
$$\begin{cases} 0 & \textit{if } i \textit{ or } j \textit{ is empty or } j \textit{ is negative infinite} \\ |i \cap_{min} j| / |cut_{j^{fK},j^{lK}}(j)| & \textit{if } i \textit{ is negative infinite} \\ (1 - D(i', E^+(j'))) \cdot D(i',j')/N'(i',j') & \textit{if } i \textit{ or } j \textit{ are positive infinite} \\ \textit{where } i' = cut_{i^{fS},max(i^{lK},j^{lK})}(i) \textit{ and } j' = cut_{j^{fS},max(i^{lK},j^{lK})}(j) & \\ (1 - D(i, E^+(j))) \cdot D(i,j)/N'(i,j) & \textit{otherwise} \end{cases}$$
*where* $N'(i,j) \stackrel{\text{def}}{=} \max_a((1 - D(shift_a(i), E^+(j))) \cdot D(shift_a(i), j))$

$$\text{starts}_{S_1,S_2,D}(i,j) \stackrel{\text{def}}{=} \begin{cases} 0 & \textit{if } i = \emptyset \textit{ or } j = \emptyset \textit{ or one of } i \textit{ and } j \textit{ is neg. infinite} \\ D(i,j) & \textit{if both } i \textit{ and } j \textit{ are negative infinite} \\ \frac{\int S_1(i)(x) \cdot S_2(j)(x)\ dx}{N(S_1(i), S_2(j))} \cdot D(i,j) & \textit{otherwise} \end{cases}$$

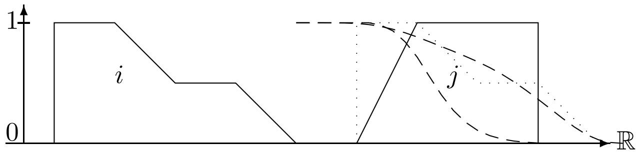
$$\text{finishes}_{F_1,F_2,D}(i,j) \stackrel{\text{def}}{=} \begin{cases} 0 & \textit{if } i = \emptyset \textit{ or } j = \emptyset \textit{ or one of } i \textit{ or } j \textit{ is pos. infinite} \\ D(i,j) & \textit{if both } i \textit{ and } j \textit{ are positive infinite} \\ \frac{\int F_1(i)(x) \cdot F_2(j)(x)\ dx}{N(F_1(i), F_2(j)))} \cdot D(i,j) & \textit{otherwise} \end{cases}$$

$$\text{equals}_D(i,j) \stackrel{\text{def}}{=} D(i,j) \cdot D(j,i)$$

*where the normalization factor is* $N(i,j) \stackrel{\text{def}}{=} \max_a \int i(x-a) \cdot j(x)\ dx$. ■

We examine the interval–interval relations now in more detail, but discuss only the finite cases. The infinite cases are explained in [5].

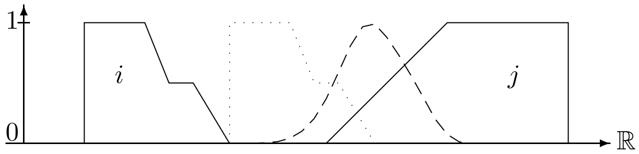
**Before**: The essential part of the definition is $\int i(x) \cdot B(j)(x)\ dx/|i|$ where $B$ is a point–interval *before*-relation. The idea of this definition is to average the point–interval *before*-relation over the interval $i$. The upper dashed line in the picture below indicates the result of the *before*-relation at position $x$ if the positive end of the interval $i$ is moved to $x$. The lower dashed line is the upper dashed line exponentiated with the exponent 10. The dotted line represents the position of the interval $i$ when the result value is dropped to 0.




before *for Fuzzy Intervals*

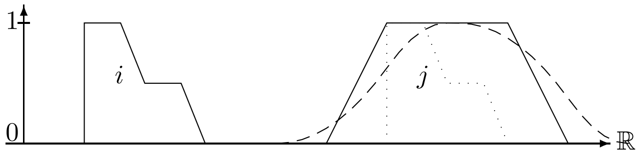
**Meets**: The classical 'meets' relation yields 'true' if the end of the first interval $i$ touches the beginning of the second interval $j$. The back end of $i$ and the front end of $j$ are therefore relevant for evaluating *meets*$(i, j)$. We can get the back end of $i$ in our fuzzy setting with the point–interval *finishes*-operator, and the front end of $j$ with the point–interval *starts*-operator. The fuzzy *meets*-relation measures how many points in the back end $F(i)$ of $i$ are in the front end $S(j)$ of $j$ and normalizes the value with the maximum possible overlap between $F(i)$ and $S(j)$.

The dashed line in the picture shows the results of the *meets*-relation when the interval $i$ is moved along the $x$-axis. The dotted figure is the position of $i$ where *meets*$(i, j)$ is maximal.




meets *for Fuzzy Intervals*

**During**: The interval–interval *during*-relation averages the point–interval *during*-relation $D_p$ by integrating over the interval $i$. The dashed line in the figure below shows the result of the *during*-relation when the *middle point* of $i$ is moved along the $x$-axis. The dotted figure indicates the leftmost position of $i$ where *during*$(i, j)$ is maximal.

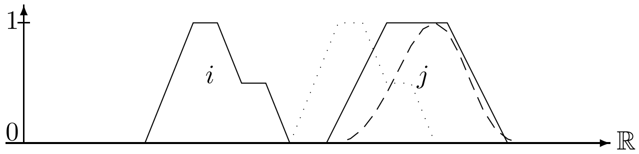



during *for Fuzzy Intervals*

**Overlaps**: The classical relation $i$ *overlaps* $j$ has two requirements: 1. a non-empty part $i_1$ of $i$ must lie before $j$, and 2. another non-empty part $i_2$ of $i$ must lie inside $j$. The first condition is encoded in the factor $1-D(i, E^+(j))$ where $D$ is a *during*-operator. $E^+(j)$ extends the rising part of $j$ to infinity. Therefore $D(i, E^+(j))$ measures the part of $i$ which is after the front part of

$j$. $1 - D(i, E^+(j))$ then measures the part of $i$ which is before the front part of $j$. This factor is multiplied with $D(i, j)$ which corresponds to the second condition. It measures to which degree $i$ is contained in $j$. The product is normalized with $\max_a((1 - D(shift_a(i), E^+(j))) \cdot D(shift_a(i), j))$ which corresponds to the maximal possible overlap when $i$ is shifted along the $x$-axis. This guarantees that there is a position for $i$ where *overlaps*$(i, j) = 1$.

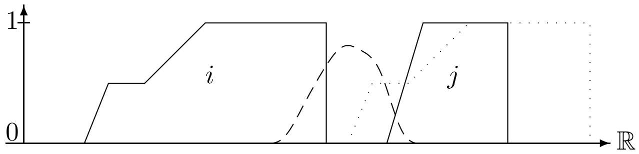
This next picture shows the result of the *overlaps*-relation where the standard *during*-operator is used.




*Example: Overlaps Relation*

The dashed line represents the result of the overlaps relation for an $x$-coordinate $x$ where the positive end of the interval $i$ is moved to $x$. The dotted figure indicates the interval $i$ moved to the position where *overlaps*$(i, j)$ becomes maximal.

**Starts** and **Finishes**: The crisp $i$ *starts* $j$-relation has two conditions: 1. the start point of $i$ and the start point of $j$ are identical, and 2. $i$ is a subset of $j$. This led to the basic definition of the fuzzy *starts*-relation as a product of the overlap between the two starting sections of $i$ and $j$ (first condition), and the *during*$(i, j)$-relation (second condition). We show the *starts*-relation between two fuzzy intervals. The dashed line is again the result of the *starts*-relation when the left end of $i$ is moved along the $x$-axis. The dotted figure shows the position of the interval $i$ where *starts*$(i, j)$ is maximal.




*Example: Starts Relation*

The fuzzy *finishes*-relation is the symmetric variant of the fuzzy *starts*-relation.

**Equals:** An interval $i$ equals an interval $j$ if $i$ is a subset of $j$ and vice versa. Therefore we get $equals_D(i, j) \stackrel{\text{def}}{=} D(i, j) \cdot D(j, i)$ where $D$ is an interval–interval *during*-operator. The picture shows the *equals*-relation for similar intervals. If $i$ is moved on top of $j$ then *equals*$(i, j) = 1$


equals *for Similar Intervals*

## 6 Summary

In this paper I proposed an operator-based approach to point–interval relations and interval–interval relations between fuzzy sets. The approach is very flexible because only the structure of the definitions of the relations is determined. The choice of the operators in the templates is left to the application. There are, however, default operators, which give quite intuitive results. The relations can produce non-trivial fuzzy values even if the intervals are crisp because they represent 'a degree of beforeness', etc. The relations are implemented in the FuTIRe library [5]. The architecture of FuTIRe is open to add other schemes, and to use them together with the crisp and operator-based schemes.

## References

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